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Terlouw, Jan

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REDUCTION OF HIGHER TYPE LEVELS BY MEANS OF AN ORDINAL ANALYSIS OF FINITE TERMS

Jan TERLOUW

Mathematical Institute, State University of Groningen, Postbus 800, 9700 AV Groningen, The Netherlands

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Introduction

In his Habilitationsschrift [6] Helmut Schwichtenberg has proved the following result about definitions of ordinal recursive functionals (a more precise formulation will be given in subsection 3.3 hereafter): if such a definition, say of functional Φ, contains subdefinitions in which auxiliary functionals of higher type levels than Φ are introduced by primitive or transfinite recursion, then these "detours through higher type levels" can be eliminated by means of transfinite recursion over a new, canonicly constructed wellordering, which has, roughly spoken, an exponentially bounded order type. Together with results of Kreisel and Tait in the other direction (see also [9]), this reveals an interesting connection between (definition by detours through) higher type levels on the one hand and transfinite recursion on the other hand.

The purpose of the present paper is to contribute to a deeper insight into this rather fundamental connection, namely by means of an alternative, conceptually more simple proof of Schwichtenberg's result. This alternative proof makes, just as Schwichtenberg's original proof (see [6] or [7]), also use of a representation of the relevant functionals by terms, but now these terms are ordinary *finite* terms, which are (technically as well as mentally) much easier to be managed than the infinite terms that play a central role in [6] and [7]. For one thing: coding finite terms by numbers is an almost trivial affair, whereas this is considerably more complicated for infinite terms (and, moreover, in order to manipulate codes of the latter one needs additional technical tools like, e.g., the primitive recursion theorem for indices of primitive recursive functions).

What now comes instead of the manipulation of infinite terms is an analysis of a certain *successor relation*, in the style of Sanchis [5] and Howard [2], between finite terms. This successor relation looks partly like a reduction relation of the familiar kind and, indeed, the original version of it was introduced by Sanchis in order to prove results, in particular strong normalization, about this notion of reduction. For the purpose of the argumentation to be presented here the following three properties of the successor relation are crucial (precise technical formulations will be given in the actual proof hereafter):

(1) For each term M the value of M can be computed in a simple, direct way from the values of its successors.

(2) The successor relation is wellfounded.

(3) The successors of a term have 'reasonably low' type levels.

As to the proofs of these properties: (3) will be trivial and (1) will be intuitively fairly clear from the definition of the successor relation; it will be a matter of routine to fill in the technical details for (1). The heart of the whole argumentation concerns (2), in fact. What will be needed for our purposes is not just an arbitrary proof of wellfoundedness, but rather a proof that is, with regard to its methods, as elementary as possible and that reaches its goal by means of a (relatively) simple ordinal assignment to terms such that each term has a bigger ordinal than its successors. Such a proof, satisfying these requirements, is presented in Section 2 of this paper. It has been strongly inspired by Howard's approach in [2] (in particular pp. 497–499, from "A more detailed examination", after Theorem 2.8). A simplification with respect to the results in [2] is that here direct ordinal assignments are constructed, independently of the notion of 'tree of a term' (cf. [2, p. 494]).

Once (the precise and sharp versions of) (1), (2) and (3) have been proved, Schwichtenberg's result can roughly be obtained as follows. Let there be given a definition of an ordinal recursive functional Φ, as at the beginning of this introduction. Then this definition can straightforwardly be transformed into a term F_Φ that represents Φ, i.e., the value (or interpretation) $[\![F_\Phi]\!]$ of F_Φ is just equal to Φ. Now it follows from (1) and (2) above that this value Φ of F_Φ can also be obtained by transfinite recursion over the successor relation. Moreover, by (3), this transfinite recursion will not go through higher type levels.

So we can replace the original definition of Φ by an equivalent definition of Φ by a transfinite recursion, without detours through higher type levels. This new definition has not yet the proper form of a definition of an ordinal recursive functional; in such a definition we can have only transfinite recursion over a wellordering of a subset of $\mathbb{N}$, the set of the natural numbers, and, moreover, the preceding values that are referred to at any stage of the recursion must be functionals of a fixed type σ. However, by coding terms and by application of so-called *valuation functionals* in the style of Schwichtenberg [6] (but somewhat simpler since all terms are finite now) it is merely a matter of routine to bring the definition into this proper form. After that the proof of the theorem will be finished; the desired (exponential) bound for the order type of the new wellordering (as mentioned above) will be clear from the constructions.

1. Functionals, terms and valuation

1.1. The set Typ of *types* is inductively defined by: $0 \in \text{Typ}$ and if $\sigma, \tau \in \text{Typ}$, then also $(\sigma \to \tau) \in \text{Typ}$. If $\sigma_1, \ldots, \sigma_n$ are types, then we write $\sigma_1 \to \cdots \sigma_{n-1} \to \sigma_n$ for $(\sigma_1 \to (\cdots (\sigma_{n-1} \to \sigma_n) \cdots))$. If $\sigma, \tau \in \text{Typ}$ and $n \in \mathbb{N}$, then $\sigma^n \to \tau$ is the type $\sigma \to \cdots \to \sigma \to \tau$ with n times σ; recursively defined: $\sigma^0 \to \tau = \tau$ and $\sigma^{n+1} \to \tau = \sigma \to \sigma^n \to \tau$. (So $\sigma^{m+n} \to \tau = \sigma^m \to \sigma^n \to \tau$.)

The *full type structure* $(\mathfrak{F}_\sigma)_{\sigma \in \text{Typ}}$ is given by: $\mathfrak{F}_0 = \mathbb{N}$ and $\mathfrak{F}_{\sigma \to \tau}$ consists of all maps from $\mathfrak{F}_\sigma$ to $\mathfrak{F}_\tau$. The elements of $\mathfrak{F}_\sigma$ are called the *functionals of type* σ. The *type level* $\text{Lev}(\sigma)$ of (a functional of) type σ is defined by:

$$\text{Lev}(0) = 0 \quad \text{and} \quad \text{Lev}(\sigma \to \tau) = \max(\text{Lev}(\sigma) + 1, \text{Lev}(\tau)).$$

(Remark: $\text{Lev}(\sigma) = \text{Lev}(\tau)$ if and only if $\mathfrak{F}_\sigma$ and $\mathfrak{F}_\tau$ have the same cardinality.) The *standard type* $\mathbf{n}$ *of level* n (with $n \in \mathbb{N}$) is defined by: $\mathbf{0} = 0$ and $\mathbf{n+1} = (\mathbf{n} \to 0)$. Clearly $\mathbf{n}$ is just the shortest type σ such that $\text{Lev}(\sigma) = n$.

In the sequel functionals (of any type) will mostly be denoted by (possibly indexed) Greek capitals $\Phi, \Psi, \Theta, \Omega$. The type may be stressed by means of an upper index: Φ^σ, etc. $\Phi \in \sigma$ means: $\Phi \in \mathfrak{F}_\sigma$. According to the standard *Schönfinkel convention* we identify $\mathfrak{F}_{\sigma_0 \to \cdots \to \sigma_k \to \tau}$ with the set of all maps from the cartesian product $\mathfrak{F}_{\sigma_0} \times \cdots \times \mathfrak{F}_{\sigma_k}$ to $\mathfrak{F}_\tau$. So instead of $\Phi(\Psi_0) \cdots (\Psi_k)$ (with $\Phi \in \sigma_0 \to \cdots \to \sigma_k \to \tau$ and $\Psi_i \in \sigma_i$, $0 \leq i \leq k$) we often write $\Phi(\Psi_0, \ldots, \Psi_k)$ (or may be just $\Phi\Psi_0 \cdots \Psi_k$). In particular, the ordinary number-theoretic functions $f : \mathbb{N}^n \to \mathbb{N}$ $(n \geq 1)$ are identified with the functionals of type $0^n \to 0$ $(n \geq 1)$, and the latter are exactly the functionals of type level 1.

The *zero functional* 0^σ of type σ is defined by: $0^0 = 0$ and if $\sigma = \sigma_0 \to \cdots \to \sigma_k \to 0$, then for all $\Psi_i \in \sigma_i$ $(0 \leq i \leq k)$, $0^\sigma(\Psi_0, \ldots, \Psi_k) = 0$.

In definitions of functionals we will sometimes make use the *meta lambda* $\lambda\!\!\!\lambda$. For example, if $\sigma = \sigma_0 \to \cdots \to \sigma_k \to 0$, then $0^\sigma = \lambda\!\!\!\lambda \Psi_0^{\sigma_0} \cdots \Psi_k^{\sigma_k} \cdot 0$.

1.2. A wellordering (*w.o.*) is in the following always (unless stated otherwise) a wellordering $\prec$ of a subset $\text{Field}(\prec)$ of $\mathbb{N}$. If $\prec$ is a w.o., then $|\prec|$ is the order type of $\prec$ and, for each number $n \in \mathbb{N}$, $|n|_\prec$ is the ordinal corresponding to n in $\prec$ (that is: the order type of the restriction of $\prec$ to $\{x \in \text{Field}(\prec) \mid x \prec n\}$; if $n \notin \text{Field}(\prec)$, then always $|n|_\prec = 0$).

From given functionals one may construct new functionals by means of *explicit definition*, *primitive recursion* or $\prec$*-recursion*, i.e., transfinite recursion over a given w.o. $\prec$. It is assumed that the reader is familiar with these notions (cf., e.g., [7, §1]). In particular, a definition of a functional $\Phi \in 0 \to \sigma$ from a functional $\Theta \in (0 \to \sigma) \to 0 \to \sigma$ by $\prec$-recursion is of the form

$$\Phi(x) = \Theta([\Phi]_{\prec x}, x),$$

where, for each $x \in \mathbb{N}$, the course-of-values functional $[\Phi]_{<x} \in 0 \to \sigma$ is defined by: $[\Phi]_{<x}(y) = \Phi(y)$ if $y < x$ and $= 0^\sigma$ otherwise.

If $n \in \mathbb{N}$, then the class PRF_n of the *primitive recursive functionals of degree n* is the smallest class (i) containing 0 and the successor function S, (ii) closed under explicit definition and (iii) closed under definition of functionals of type level $\leq n$ by means of primitive recursion. If, in addition, $<$ is a w.o., then the class $\mathrm{REC}_n(<)$ of the *$<$-recursive functionals of degree n* is the smallest class with properties (i), (ii), (iii) and (iv) closed under definition of functionals of type level $\leq n$ by means of $<$-recursion (cf. [9]).

We write

$$\mathrm{PRF} = \bigcup_{n \in \mathbb{N}} \mathrm{PRF}_n \quad \text{and} \quad \mathrm{REC}(<) = \bigcup_{n \in \mathbb{N}} \mathrm{REC}_n(<).$$

The set of the primitive recursive functions in the ordinary, well-known sense will be denoted by Prf. From Prf we define: EXP is the smallest class of functionals such that (i) $0 \in \mathrm{EXP}$ and $\mathrm{Prf} \subset \mathrm{EXP}$ and (ii) EXP is closed under explicit definition. It is clear that $\mathrm{EXP} \subseteq \mathrm{PRF}_1$. In fact, it can even be proved that $\mathrm{EXP} = \mathrm{PRF}_1$ (this follows from Pétèr [4] or Tait [8]), but we will not need this stronger result.

Lemma. *Let $<$ be a w.o. and let $K_<$ be the characteristic function of $<$. Then $K_< \in \mathrm{REC}_1(<)$.*

Proof. Define the function $f : \mathbb{N}^2 \to \mathbb{N}$ by: $f(x, y) = 1$ if $x = y$ and $= [f]_{<x}(y, y)$ otherwise; then $f \in \mathrm{REC}_1(<)$ and $y < x \leftrightarrow (y \neq x \wedge f(x, y) = 1)$. $\square$

1.3. We now define a system TR of terms for the representation of the $<$-recursive functionals from REC($<$), where $<$ is any fixed w.o. We will make use of some basic notions from the *typed lambda calculus*; it is assumed that the reader is familiar with these.

TR-terms (in the sequel simply *terms*) will be built up from the following symbols: typed variables (infinitely many $v_0^\sigma, v_1^\sigma, v_2^\sigma, v_3^\sigma, \ldots$ for every type σ), brackets and $\lambda, \bar{0}, \bar{S}, R, T, [T]$.

The set $\mathrm{TR} = \bigcup \{\mathrm{TR}(\sigma) \mid \sigma \in \mathrm{Typ}\}$ of the (TR-) terms is inductively defined as follows; here $\bar{0}, \bar{1}, \bar{2}, \ldots$ are the *numerals*, with $\bar{0} \equiv \bar{0}$ and $\overline{n+1} \equiv (\bar{S}\bar{n})$, and FV($M$) is the set of the *free variables* of term M, in the usual sense.

1. $\bar{0} \in \mathrm{TR}(0)$, $\quad \bar{S} \in \mathrm{TR}(0 \to 0)$.
2. $v_i^\sigma \in \mathrm{TR}(\sigma)$ for all $\sigma \in \mathrm{Typ}$, $i \in \mathbb{N}$.
3. $M \in \mathrm{TR}(\sigma \to \tau)$, $N \in \mathrm{TR}(\sigma) \Rightarrow (MN) \in \mathrm{TR}(\tau)$.
4. $M \in \mathrm{TR}(\tau) \Rightarrow (\lambda v_i^\sigma M) \in \mathrm{TR}(\sigma \to \tau)$.
5. $G \in \mathrm{TR}(\sigma \to 0 \to \sigma)$, $H \in \mathrm{TR}(\sigma)$ with $\mathrm{FV}(G) = \mathrm{FV}(H) = \emptyset \Rightarrow (\mathrm{R}GH) \in \mathrm{TR}(0 \to \sigma)$.
6. $G \in \mathrm{TR}((0 \to \sigma) \to 0 \to \sigma)$ with $\mathrm{FV}(G) = \emptyset \Rightarrow (\mathrm{T}G) \in \mathrm{TR}(0 \to \sigma)$ and $([T]G\bar{n}) \in \mathrm{TR}(0 \to \sigma)$ for all $n \in \mathbb{N}$.

The terms of the form (RGH), (TG) or ($[T]G\bar{n}$) will be called *recursion blocks*.

The elements of $\mathrm{TR}(\sigma)$ are the *terms of type* σ. If M is a term, then we denote by $\mathrm{Typ}(M)$ and $\mathrm{Lev}(M)$ the type and the type level of (the type of) M, respectively. In addition we define: the *degree* $\mathrm{Deg}(M)$ of M is the maximum of the type levels of the recursion blocks occurring in M. (If M contains no recursion blocks, then $\mathrm{Deg}(M)=0$.)

For each $n\in\mathbb{N}$ we define:

$$\mathrm{TR}_n=\{M\in\mathrm{TR}\mid \mathrm{Deg}(M)\leqslant n\}.$$

So TR_0 consists just of the terms that are generated by clauses 1, 2, 3, 4 only. These are the *pure typed lambda terms*. The set of these is also denoted by λ^{τ}. (So we have: $\mathrm{TR}_0=\lambda^{\tau}$.)

Notations. In general, variables will be denoted by X, Y, Z, x, y, z, terms by A, B, F, G, H, K, L, M, N and, in particular, terms of type 0 by s, t. As usual we write $MN_1\cdots N_k$ (or $M\vec{N}$) instead of $(\cdots((MN_1)N_2)\cdots N_k)$ and $\lambda x_1\cdots x_k\cdot M$ instead of $(\lambda x_1(\lambda x_2\cdots(\lambda x_k M)\cdots))$.

Var is the set of all variables and $\mathrm{Var}(\sigma)$ is the ordered set of all variables of type σ $(v_0^\sigma, v_1^\sigma, v_2^\sigma,\ldots)$. If V is any subset of TR, then we write, for each type σ,

$$V(\sigma)=\{M\in V\mid \mathrm{Typ}(M)=\sigma\}\quad(=\mathrm{TR}(\sigma)\cap V).$$

1.4. Terms are interpreted as follows, using the notion of *assignment*. By definition, an assignment is a map ρ such that the domain $\mathrm{dom}(\rho)$ of ρ is a finite subset of Var and for each $x\in\mathrm{dom}(\rho)$, $\rho(x)$ is a functional of the same type as x. If ρ is an assignment and if $x_1,\ldots,x_k$ are distinct variables of the same types as the functionals $\Phi_1,\ldots,\Phi_k$ respectively, then $\rho[\Phi_1,\ldots,\Phi_k/x_1,\ldots,x_k]$ is the following assignment ρ':

$$\mathrm{dom}(\rho')=\mathrm{dom}(\rho)\cup\{x_1,\ldots,x_k\},\qquad \rho'(x_i)=\Phi_i\quad(1\leqslant i\leqslant k)\quad\text{and}$$
$$\rho'(y)=\rho(y)\quad\text{for all } y\in\mathrm{dom}(\rho)\setminus\{x_1,\ldots,x_k\}.$$

An assignment *for* a term $M\in\mathrm{TR}$ is an assignment ρ such that $\mathrm{FV}(M)\subseteq\mathrm{dom}(\rho)$.

Now let $<$ be a fixed w.o. Then the value or interpretation $[\![M]\!]_\rho\in\mathfrak{F}_\sigma$ of a term $M\in\mathrm{TR}(\sigma)$ relative to any assignment ρ for M is defined as follows, by recursion on the length of M.

1. $[\![\bar{0}]\!]_\rho=0,\qquad [\![\bar{S}]\!]_\rho=S.$
2. $[\![x]\!]_\rho=\rho(x)$ for $x\in\mathrm{dom}(\rho)$.
3. $[\![MN]\!]_\rho=[\![M]\!]_\rho([\![N]\!]_\rho)$.
4. $[\![\lambda xM]\!]_\rho=\lambda\!\!\!\lambda\Psi^\sigma\cdot[\![M]\!]_{\rho[\Psi/x]}$, where $\sigma=\mathrm{Typ}(x)$.
5. $[\![\mathrm{RGH}]\!]_\rho$ is the functional Φ that is defined from $[\![G]\!]_\rho$ and $[\![H]\!]_\rho$ by primitive recursion: $\Phi(0)=[\![H]\!]_\rho$ and $\Phi(x+1)=[\![G]\!]_\rho(\Phi(x),x)$.
6. $[\![\mathrm{TG}]\!]_\rho$ is the functional Φ that is defined from $[\![G]\!]_\rho$ by $<$-recursion, as in 1.2. And for each $n\in\mathbb{N}$, $[\![[T]G\bar{n}]\!]_\rho$ is the course-of-values $[\Phi]_{<n}$ of this Φ below n.

Clearly, as to the assignment ρ for $M, [\![M]\!]_\rho$ depends only on the restriction $\rho \restriction \mathrm{FV}(M)$ of ρ to $\mathrm{FV}(M)$. So if M is closed, i.e., $\mathrm{FV}(M) = \emptyset$, then we simply write $[\![M]\!]$ instead of $[\![M]\!]_\rho$.

If desirable, the w.o. $\prec$ may be made explicit by means of an upper index: $[\![M]\!]_\rho^\prec, [\![M]\!]^\prec$.

Lemma. *Let* $\prec$ *be a w.o. Then for each* $n \in \mathbb{N}$:

$$\mathrm{REC}_n(\prec) = \{[\![M]\!]^\prec \mid M \in \mathrm{TR}_n \text{ and } \mathrm{FV}(M) = \emptyset\}.$$

Proof. Obvious. (As to $\supseteq$: make use of $K_\prec \in \mathrm{REC}_1(\prec)$, see Lemma 1.2, in order to deal with $[T]G\bar{n}$.) $\square$

2. The successor relation and ordinal assignments

In this section we define and investigate, purely from the syntactical point of view, the *successor relation* $\to$ between terms. The three crucial properties of this relation have already been roughly indicated in the introduction; in this section we will be concerned with the second one: wellfoundedness. In particular we will construct so-called *reducing ordinal assignments* to terms, satisfying the condition that each term obtains a bigger ordinal than any of its successors.

We will also (and, in fact, mainly) be concerned with so-called *weakly reducing ordinal assignments*, that satisfy certain weaker versions of the above condition. The use of these will be rather essential, namely for the following two reasons.

(1) Weakly reducing ordinal assignments will serve as intermediate stages in the constructions of reducing ordinal assignments.

(2) In the applications in the next section, Section 3, it will in general not only be the case that a (suitably chosen) weakly reducing ordinal assignment is already good enough, but also that this is preferable to the available 'strongly reducing' ordinal assignment because of the fact that it assigns considerably smaller ordinals (which is important in order to get low bounds for the order types of the new wellorderings, as mentioned at the beginning of the introduction).

Convention. In the rest of this section $\prec$ is a fixed w.o. with infinite order type $|\prec|$. Instead of $|n|_\prec$ (where $n \in \mathbb{N}$) we will simply write $|n|$ (cf. 1.2).

2.1. We start with introducing some more syntactical notions, in addition to those of 1.3.

As usual, $F[G_1, \ldots, G_k/X, \ldots, X_k]$ (with $X_1, \ldots, X_k$ distinct variables of the same type as the terms $G_1, \ldots, G_k$ respectively) is the result of *simultaneous substitution* of $G_1, \ldots, G_k$ for the free occurrences of $X_1, \ldots, X_k$ in F. (*Remark.* Here it is understood that bound variables are renamed whenever necessary. We will not dwell on this matter; in fact we identify tacitly any two terms that are equal up to *α-conversion*; cf. Barendregt [1, Appendix C].)

$L \cong M$ means that L is a *variant* of M; here 'variant of' is, by definition, the smallest equivalence relation on TR with the following property: if L can be obtained from M by replacing a free occurrence of a variable X by a variable Y of the same type such that resulting occurrence of Y in L is still free, then L is a variant of M.

$L \subset_{\mathrm{arg}} M$ means that L is an *argument* of M, i.e., we can write $M \equiv FK_1 \cdots K_k$, with $F, K_1, \ldots, K_k \in \mathrm{TR}$, such that $L \in \{K_1, \ldots, K_k\}$.

$L \subset_{\mathrm{h}} M$ means that L is a *head term* of M, i.e., we can write $M \equiv LK_1 \cdots K_k$ with $k \geq 1$.

$L \subset M$ means that L is a *subterm* of M, in the usual sense. (So in particular: $M \subset M$; the subterms of M other than M are called the *proper substerms* of M.) The set of all subterms of M is denoted by $\mathrm{Sub}(M)$.

The *hereditary type level* $\mathrm{Lev}^*(M)$ of term M is defined by:

$$\mathrm{Lev}^*(M) = \max\{\mathrm{Lev}(L) \mid L \subset M\}.$$

Recall that λ^τ $(= \mathrm{TR}_0)$ is the set of the pure typed lambda terms (cf. 1.3). Now we define for each $n \in \mathbb{N}$:

$$\lambda_n^\tau = \{M \in \lambda^\tau \mid \mathrm{Lev}^*(M) \leq n\}.$$

2.2. The *successor relation* $\rightarrow \subset \mathrm{TR} \times \mathrm{TR}$ is defined as follows (instead of $(M, N) \in \rightarrow$ we will write $M \rightarrow N$): if $M, N \in \mathrm{TR}$, then $M \rightarrow N$ holds if and only if this is forced by one of the clauses (1), (2), ..., (7*b*) below, where, in addition, it is required that *in clauses* (2), (3), ..., (7*b*) *the term at the left hand side of* $\rightarrow$ *has type* 0.

The terms N such that $M \rightarrow N$ (also written: $N \leftarrow M$) will be called the *successors* of M.

(1) $F \rightarrow FX_1 \cdots X_k$ if $\mathrm{Type}(F) \neq 0$ and $X_1, \ldots, X_k$ are variables such that $FX_1 \cdots X_k$ is a term of type 0.

(2) $AK_1 \cdots K_k \rightarrow K_i$ $(1 \leq i \leq k)$ if A is a variable or the successor $\bar{S}$.

(3) $(\lambda XG)HK_1 \cdots H_k \rightarrow G[H/X]K_1 \cdots K_k$.

(4) $BtK_1 \cdots K_k \rightarrow t, B\bar{0}K_1 \cdots K_k, B\bar{1}K_1 \cdots K_k, \ldots, B\bar{n}K_1 \cdots K_k, \ldots$ if B is a recursion block and t is not a numeral.

(5a) $RGH\bar{0}K_1 \cdots K_k \rightarrow HK_1 \cdots K_k$.

(5b) $RGH\overline{n+1}K_1 \cdots K_k \rightarrow G(RGH\bar{n})\bar{n}K_1 \cdots K_k$.

(6) $TG\bar{n}K_1 \cdots K_k \rightarrow G([T]G\bar{n})\bar{n}K_1 \cdots K_k$.

(7a) $[T]G\bar{n}\bar{m}K_1 \cdots K_k \rightarrow TG\bar{m}K_1 \cdots K_k$ if $m \prec n$.

(7b) $[T]G\bar{n}\bar{m}K_1 \cdots K_k \rightarrow \bar{0}$ if $\neg m \prec n$.

2.3. In the following we will mean by an *ordinal assignment* exclusively a map $|\cdot| : V \rightarrow \mathrm{ON}$ (with ON the class of all ordinals) such that

1. the domain V is one of the classes $\mathrm{TR}, \mathrm{TR}_n, \lambda^\tau, \lambda_n^\tau$ $(n \in \mathbb{N})$, and

2. the map 'identifies variants', i.e.,

$$\forall M \in V\, \forall L \cong M\, |L| = |M|.$$

Such an ordinal assignment $|\cdot|: V \to \mathrm{ON}$ is called *reducing* if it satisfies the following conditions:

(Suc) $\forall M \in V\, \forall L \leftarrow M\, |L| < |M|$

(Arg) $\forall M \in V\, \forall L \subset_{\mathrm{arg}} M\, |L| < |M|$

(Fex) $\forall M \in V(\sigma \to \tau)\, \forall X \in \mathrm{Var}(\sigma)\, [|MX| \leqslant |M| \wedge (\tau = 0 \Rightarrow |MX| < |M|)]$.

(Here 'Fex' refers to 'free extension'.)

It is called *weakly reducing* if it satisfies (Arg), (Fex) and the following weaker version of (Suc):

(Suh) $\forall M \in V\, [(\forall L \leftarrow M\, |L| < |M|) \vee \exists F \subset_{\mathrm{h}} M\, |F| < |M|]$.

The following notion of 'rank' (corresponding to Howard's notion of 'degree' in [2, p. 494]) determines a hierarchy within the class of all weakly reducing ordinal assignments:

Let $0 \leqslant p \leqslant \omega$. An *ordinal assignment of rank* p is, by definition, an ordinal assignmant $|\cdot|: V \to \mathrm{ON}$ that satisfies the conditions (Arg), (Fex) and

$$(\mathrm{Suh}_p)\ \forall M \in V\, [(\forall L \leftarrow M\, |L| < |M|) \vee \exists F \subset_{\mathrm{h}} M\, (|F| < |M| \wedge \mathrm{Lev}(F) \leqslant p)].$$

Some trivial observations:

(1) The ordinal assignments of rank 0 are exactly the reducing ordinal assignments and those of rank ω the weakly reducing ordinal assignments.

(2) Suppose $0 \leqslant p \leqslant q \leqslant \omega$. Then every ordinal assignment of rank p is also an ordinal assignment of rank q.

(3) Every weakly reducing ordinal assignment with domain $V = \lambda_n^\tau$ is automatically an ordinal assignment of rank n (just because of the fact that every term in λ_n^τ has type level $\leqslant n$).

Before we turn to the actual construction of (weakly) reducing ordinal assignments, we now present first (in subsections 2.4–2.14) a general, rather simple method for transforming ordinal assignments of rank > 0 into ordinal assignments of smaller rank (still with the same domain V). This method has been inspired by the approach of Howard in [2, pp. 493–499]. It will enable us afterwards, in Section 3 (to be more specific: in Definition 3.6(ii)), to replace (certain restrictions of) the weakly reducing ordinal assignment $|\cdot|_{\mathrm{R}}: \mathrm{TR} \to \mathrm{ON}$, that will be constructed in 2.18, by ordinal assignments of sufficiently low ranks. The operations that perform this lowering of ranks will be denoted by RED_p (see Theorem 2.6). (By the way, these operations play also, at an earlier stage, a role in the construction of $|\cdot|_{\mathrm{R}}$ itself, namely in order to obtain the auxiliary ordinal assignment $|\cdot|_\lambda : \lambda^\tau \to \mathrm{ON}$ in 2.16, Construction IB.)

2.4. Let M be a term. Loosely speaking we mean by a *frame* of M a term L that can be obtained from M by replacing some (possibly none) subterms by single (not necessarily distinct), free variables. More precisely: a frame of M is a term L

such that

$$L \cong M_0 \quad \text{and} \quad M \equiv M_0[N_1, \ldots, N_k/X_1, \ldots, X_k]$$

for some term M_0 and some (possibly empty) sequences $\vec{N} = (N_1, \ldots, N_k)$ and $\vec{X} = (X_1, \ldots, X_k)$ of terms, respectively variables, where N_i and X_i have the same type $(1 \leqslant i \leqslant k)$.

Note that from this more precise definition it follows that the occurrences of subterms of M that are replaced by single variables may not contain occurrences of variables that are bound from the outside.

Notation. If L and M are terms, then $L \sqsubset M$ means that L is a frame of M.

Without proof we mention the following elementary fact:

Lemma. *Supppose*: $M \equiv M_0[F_1, \ldots, F_k/X_1, \ldots, X_k] \equiv N_0[G_1, \ldots, G_m/Y_1, \ldots, Y_m]$, *where* $X_1, \ldots, X_k \in \mathrm{FV}(M_0)$ *and* $Y_1, \ldots, Y_m \in \mathrm{FV}(N_0)$. *Suppose also*: $M_0 \cong N_0$. *Then*

$$\{F_1, \ldots, F_k\} \backslash \mathrm{Var} = \{G_1, \ldots, G_m\} \backslash \mathrm{Var}.$$

Proof. By a routine inspection of the syntactical structure of terms. □

This lemma enables us to define unambiguously: if $L \sqsubset M$, then the subset $\mathrm{St}(L, M)$ of $\mathrm{Sub}(M)$ is defined by: write

$$M \equiv M_0[N_1, \ldots, N_k/X_1, \ldots, X_k] \quad \text{with } L \cong M_0;$$

then

$$\mathrm{St}(L, M) = \{N_i \mid 1 \leqslant i \leqslant k \wedge X_i \in \mathrm{FV}(M_0) \wedge N_i \notin \mathrm{Var}\}.$$

The elements of $\mathrm{St}(L, M)$ might be called: the *substitution terms for filling up the frame L to* (a variant of) M.

If M is a term and $0 \leqslant p \leqslant \omega$, then a *p-frame* of M is a frame L of M such that

$$\forall N \in \mathrm{St}(L, M)\ \mathrm{Lev}(N) \leqslant p.$$

Notation. $L \sqsubset_p M$ means that L is a p-frame of M.

2.5. Definition. (i) Let $|\cdot| : V \to \mathrm{ON}$ be an ordinal assignment. Let $0 \leqslant p \leqslant \omega$. Then $\mathrm{SUB}_p(|\cdot|)$ is the following map $\|\cdot\| : V \to \mathrm{ON}$, which is defined by recursion on the length $\mathrm{lh}(M)$ of terms $M \in V$:

$$\|M\| = \min\{\|\mathrm{St}(L, M)\| + |L| \mid L \sqsubset_p M \wedge M \notin \mathrm{St}(L, M)\}$$

where, in general for any map $\|\cdot\| : V \to \mathrm{ON}$ and any finite subset $P \subset V$, $\|P\| = \max\{\|N\| \mid N \in P\}$ $(=0$ if $P = \emptyset)$.

Remark. If $L \sqsubset M$, then the condition $M \notin \mathrm{St}(L, M)$ means that L is a proper frame of M in the following sense: if $M \notin \mathrm{Var}$, then $L \notin \mathrm{Var}$.

It is easily seen that this map $\|\cdot\|$ is an ordinal assignment again (in the sense of

2.3); see Lemma 2.11(i). So we have defined a map

$$\mathrm{SUB}_{\mathrm{p}}:\mathrm{ASS}\to\mathrm{ASS}$$

where, by definition, ASS is the class of all ordinal assignments.

(ii) Let $\exp:\mathrm{ON}\to\mathrm{ON}$ be exponentiation with respect to base 2; $\exp(\alpha)=2^{\alpha}$. For each p $(0\leqslant p\leqslant\omega)$ we define as follows an operation

$$\mathrm{RED}_{\mathrm{p}}:\mathrm{ASS}\to\mathrm{ASS}.$$

Let $|\cdot|:V\to\mathrm{ON}$ be an ordinal assignment. First define:

$$\exp(|\cdot|):V\to\mathrm{ON}:\quad M\mapsto\exp(|M|);$$

obviously this is an ordinal assignment again. Next put;

$$\mathrm{RED}_{\mathrm{p}}(|\cdot|)=\mathrm{SUB}_{\mathrm{p}}(\exp(|\cdot|)).$$

Now the intended result about lowering the rank of weakly reducing ordinal assignments is the following theorem:

2.6. Theorem. *Let $|\cdot|:V\to\mathrm{ON}$ be an ordinal assignment of rank $p+1$, where $p<\omega$. Let $\|\cdot\|=\mathrm{RED}_{\mathrm{p}}(|\cdot|)$: $V\to\mathrm{ON}$. Then $\|\cdot\|$ is an ordinal assignment of rank p and $\forall M\in V$ $\|M\|\leqslant\exp(|M|)$.*

Proof. See 2.12.

By repeated application of this theorem it is possible to transform any weakly reducing ordinal assignment $|\cdot|:V\to\mathrm{ON}$ ultimately into an ordinal assignment $\|\cdot\|:V\to\mathrm{ON}$ of rank 0. This is done as follows (see 2.7 and 2.8):

2.7. Definition. By recursion on $n-p$ we define as follows an operation

$$\mathrm{RED}_{n,\mathrm{p}}:\mathrm{ASS}\to\mathrm{ASS}$$

for each pair (n,p) with $p\leqslant n<\omega$. $\mathrm{RED}_{n,n}$ is the identity: $\mathrm{ASS}\to\mathrm{ASS}$ and if $p<n$, then $\mathrm{RED}_{n,\mathrm{p}}$ is the composition $\mathrm{RED}_{\mathrm{p}}\circ\mathrm{RED}_{n,\mathrm{p}+1}$. So if $p<n$, then $\mathrm{RED}_{n,\mathrm{p}}$ is obtained by iterated composition:

$$\mathrm{RED}_{n,\mathrm{p}}=\mathrm{RED}_{\mathrm{p}}\circ\mathrm{RED}_{\mathrm{p}+1}\circ\cdots\circ\mathrm{RED}_{n-1}.$$

Theorem. *Let $|\cdot|:V\to\mathrm{ON}$ be an ordinal assignment of rank $n<\omega$. Let $p\leqslant n$ and let $\|\cdot\|=\mathrm{RED}_{n,\mathrm{p}}(|\cdot|)$: $V\to\mathrm{ON}$. Then $\|\cdot\|$ is an ordinal assignment of rank p and $\forall M\in V$ $\|M\|\leqslant 2_{n-\mathrm{p}}(|M|)$, where, by definition $2_0(\alpha)=\alpha$ and $2_{k+1}(\alpha)=\exp(2_k(\alpha))$ for $\alpha\in\mathrm{ON}$, $k\in\mathbb{N}$.*

Proof. From Theorem 2.6, by induction on $n-p$. $\square$

2.8. Definition. Let $|\cdot|$ be an ordinal assignment: $V\to\mathrm{ON}$, where $V=\lambda^{\tau}$ or

$V = \mathrm{TR}$. Then the ordinal assignment $\mathrm{RED}_*(|\cdot|): V \to \mathrm{ON}$ is defined by

$$\mathrm{RED}_*(|\cdot|)(M) = \min\{\mathrm{RED}_{n,0}(|\cdot|)(M) \mid h(M) \le n < \omega\}$$

where $h(M) = \mathrm{Lev}^*(M)$ (see 2.1) if $V = \lambda^\tau$ and $h(M) = \mathrm{Deg}(M)$ (see 1.3) if $V = \mathrm{TR}$.

Theorem. *Let* $(V, h) = (\lambda^\tau, \mathrm{Lev}^*)$ *or* $(V, h) = (\mathrm{TR}, \mathrm{Deg})$. *Let* $|\cdot|: V \to \mathrm{ON}$ *be a weakly reducing ordinal assignment such that, for each* $n \in \mathbb{N}$, *the restriction of* $|\cdot|$ *to* $V_n = \{M \in V \mid h(M) \le n\}$ *is of rank* n. (*Note that* $V_n = \lambda^\tau_n$ *or* $V_n = \mathrm{TR}_n$.)

Then the ordinal assignment $\mathrm{RED}_*(|\cdot|): V \to \mathrm{ON}$ *is reducing* (*in other words*: *of rank* 0) *and* $\forall M \in V$ $\mathrm{RED}_*(|\cdot|)(M) \le 2_{h(M)}(|M|)$.

Remark. If $V = \lambda^\tau$ and if $|\cdot|: V \to \mathrm{ON}$ is weakly reducing, then it follows automatically that for each $n \in \mathbb{N}$ the restriction of $|\cdot|$ to V_n $(= \lambda^\tau_n)$ is of rank n; see observation (3) in 2.3.

Proof. For each $n \in \mathbb{N}$, let $|\cdot|_n : V_n \to \mathrm{ON}$ be the restriction of $|\cdot|$ to V_n, which is, by hypothesis, of rank n. Write $\|\cdot\|_n = \mathrm{RED}_{n,0}(|\cdot|_n): V_n \to \mathrm{ON}$. Then, by Theorem 2.7 above (with $p = 0$), $\|\cdot\|_n$ is of rank 0 and $\|M\|_n \le 2_n(|M|_n) = 2_n(|M|)$ for each $M \in V_n$. From this it follows easily that the map $\|\cdot\|: V \to \mathrm{ON}$ defined by $\|M\| = \min\{\|M\|_n \mid n \in \mathbb{N}, M \in V_n\}$ is also an ordinal assignment of rank 0. Moreover: if $M \in V$ and $n = \mathrm{lh}(M)$, then in particular $M \in V_n$ and therefore $\|M\| \le \|M\|_n \le 2_n(|M|)$. On the other hand it is clear from the definitions that this map $\|\cdot\|$ is exactly $\mathrm{RED}_*(|\cdot|)$. So we are done. □

It is noteworthy that in order to compute the ordinal $\mathrm{RED}_*(|\cdot|)(M)$, as defined above, we only need to know the values of $\mathrm{RED}_{n,0}(|\cdot|)(M)$ for $h(M) \le n \le \mathrm{Lev}^*(M) + 1$ rather than for all n with $h(M) \le n < \omega$. This is expressed by the following lemma:

Lemma. *Let* $(V, h) = (\lambda^\tau, \mathrm{Lev}^*)$ *or* $(V, h) = (\mathrm{TR}, \mathrm{Deg})$. *Let* $|\cdot|: V \to \mathrm{ON}$ *be an ordinal assignment. Then for all* $M \in V$:

$$\mathrm{RED}_*(|\cdot|)(M) = \min\{\mathrm{RED}_{n,0}(|\cdot|)(M) \mid h(M) \le n \le \mathrm{Lev}^*(M) + 1\}.$$

Proof. See 2.14.

2.9. We will now present the proofs that have been omitted above; these will occupy the subsections 2.9–2.14. The reader who is not interested in them (and satisfies himself instead with the knowledge of the rather simple constructions as presented in 2.5–2.8) may prefer to turn immediately to the second part of this section, namely subsections 2.15–2.19, which will be devoted to the actual construction of ordinal assignments $|\cdot|_\lambda$ and $|\cdot|_R$.

We start with defining two special kinds of frames and will prove thereafter a crucial lemma, which expresses that in some way the operation of taking frames of a term commutes with the operations of taking arguments or successors of terms.

A *critical frame* of term M is a frame L of M such that $\mathrm{Typ}(M)=0$ and we can write $L\equiv XK_1\cdots K_k$ and $M\equiv FL_1\cdots L_k$, where $k\geqslant 0$, X is a variable and F is a term that is not a variable.

An *improper frame* of M is a frame L of M such that $\mathrm{Typ}(M)=0$ and we can write $L\equiv BtK_1\cdots K_k$ and $M\equiv B\bar{n}L_1\cdots L_k$, where B is a recursion block, $n\in\mathbb{N}$ and t is a term that is not a numeral.

Lemma. (i) *Suppose*: $L\sqsubset M$ *and* $M'\subset_{\arg}M$. *Then*

$$[\exists L'\subset_{\arg}L\,(L'\sqsubset M'\wedge \mathrm{St}(L',M')\subseteq \mathrm{St}(L,M))]\vee\exists N\in\mathrm{St}(L,M)\; M'\subset_{\arg}N.$$

(ii) *Suppose*: $L\sqsubset M$ *and* L *is neither critical nor improper* (*w.r.t.* M). *Then*

$$\forall M'\leftarrow M\,\exists L'\leftarrow L\; L'\sqsubset M'\wedge\mathrm{St}(L',M')\subseteq\mathrm{St}(L,M).$$

Proof. Routine; first write, for any given frame L of M, $M\equiv M_0[N_1,\ldots,N_k/X_1,\ldots,X_k]$ with $M_0\cong L$ and $X_1,\ldots,X_k\in\mathrm{FV}(M_0)$, and next $M_0\equiv \mathrm{F}K_1\cdots K_m$, where $m\geqslant 0$ and the term F is not of the form F_1F_2 (i.e., F has no arguments). Then $M\equiv \mathrm{G}L_1\cdots L_m$ with $G\equiv F[\vec{N}/\vec{X}]$, $L_i\equiv K_i[\vec{N}/\vec{X}]$ $(1\leqslant i\leqslant m)$, and $L\equiv F'K_1'\cdots K_m'$ with $F'\cong F$, $K_i'\cong K_i$ $(1\leqslant i\leqslant m)$. After this the results (i) and (ii) follow by a simple inspection of definitions. As to (i): observe: if $M'\subset_{\arg}M$, then (1) $M'\in\{L_1,\ldots,L_m\}$ or (2) $M'\subset_{\arg}G$ and $G\in\{N_1,\ldots,N_k\}\setminus\mathrm{Var}=\mathrm{St}(L,M)$. As to (ii): observe in particular (by inspection of the clauses (2), (3), …, (7b) in 2.2): if $\mathrm{Typ}(M)=0$ and L is neither critical nor improper (w.r.t. M), then $\forall M'\leftarrow M\,\exists M_0'\leftarrow M_0\;\; M'\equiv M_0'[\vec{N}/\vec{X}]$('$\rightarrow$ *commutes with substitution*'). □

2.10. We will also need some more facts about the 'frame of' relation $\sqsubset$. The proof of these facts will be omitted, since they can easily be obtained by means of a routine inspection of the syntactical structure of terms (just as in the case of Lemma 2.4).

Lemma. (i) *The 'frame of' relation* $\sqsubset$ *is a partial order on* TR *modulo* $\cong$; *that is*: *for all terms* K, L, M: $(L\sqsubset M\wedge M\sqsubset L)\Leftrightarrow L\cong M$ *and* $(K\sqsubset L\wedge L\sqsubset M)\Rightarrow K\sqsubset M$. *Moroever*: *if* $L\cong M$, *then* $\mathrm{St}(L,M)=\emptyset$ *and if* $K\sqsubset L\sqsubset M$, *then*

$$\forall N\in\mathrm{St}(K,M)\,[N\in\mathrm{St}(L,M)\vee\exists F\in\mathrm{St}(K,L)\;(F\sqsubset N\wedge\mathrm{St}(F,N)\subseteq\mathrm{St}(L,M))].$$

(ii) *Suppose*: $K\sqsubset M$ *and* $L\sqsubset M$. *Then there exists an infimum* $\inf(K,L)$ *of* K *and* L *with respect to* $\sqsubset$. *Moreover*: *this infimum* (*which is unique up to* $\cong$) *satisfies*:

$$\mathrm{St}(\inf(K,L),M)\subseteq\mathrm{St}(K,M)\cup\mathrm{St}(L,M)\quad\textit{and}$$

$$\forall N\in\mathrm{St}(\inf(K,L),K)\,\exists F\in\mathrm{St}(L,M)\;N\sqsubset F.\qquad\square$$

2.11. Lemma. *Let* $|\cdot|: V \to \mathrm{ON}$ *be an ordinal assignment and let* $\|\cdot\| = \mathrm{SUB}_p(|\cdot|)$, *where* $0 \leq p \leq \omega$. *Then the following hold:*

(i) $\|\cdot\|$ *is an ordinal assignment again (in the sense of* 2.3).

(ii) $\|\cdot\| \leq |\cdot|$ (*that is*: $\forall M \in V\, \|M\| \leq |M|$).

(iii) $\forall M \in V\, \forall L \sqsubset_p M\, \|M\| \leq \|\mathrm{st}(L, M)\| + \|L\|$.

(iv) *Suppose that the ordinal assignment* $|\cdot|$ *is weakly reducing. Then* $\|\cdot\|$ *is also weakly reducing. Moreover: let* $M \in V$ *and let* $\|M\| = \|\mathrm{St}(L, M)\| + |L|$ *with* $L \sqsubset_p M$ *and* $M \notin \mathrm{St}(L, M)$; *then:*

(1) *If* $\forall L' \leftarrow L\ |L'| < |L|$ *and* L *is not critical (w.r.t.* M), *then also* $\forall M' \leftarrow M\, \|M'\| < \|M\|$.

(2) *If* L *is a critical frame of* M, *then* $\exists G \subset_h M\ (\|G\| < \|M\| \wedge \mathrm{Lev}(G) \leq p)$.

(3) $\forall F \subset_h L\ [|F| < |L| \Rightarrow \exists G \subset_h M\ (\|G\| < \|M\| \wedge \mathrm{Lev}(G) = \mathrm{Lev}(F))]$.

In consequence: if $|\cdot|$ *is of rank* q, *then* $\|\cdot\|$ *is of rank* $\max(p, q)$.

(v) *Suppose*: $p < \omega$ *and* $|\cdot|$ *is of rank* $p+1$. *Moreover suppose*: $\forall K, L \in V$ $(|K| < |L| \Rightarrow |K| + |K| \leq |L|)$. *Then* $\|\cdot\|$ *is of rank* p.

Proof. (i) We must prove that $\|\cdot\|$ identifies variants. Because of the symmetry of $\cong$ it suffices to prove $\forall M \in V\, \forall M' \cong M\, \|M'\| \leq \|M\|$. But this can easily be done by induction on $\mathrm{lh}(M)$. (Observe: $L \sqsubset M \cong M' \Rightarrow \exists L' \sqsubset M'\, \forall N' \in \mathrm{St}(L', M')\ \exists N \in \mathrm{St}(L, M)\ N' \cong N$.)

(ii) Obvious, since, for all terms M, $M \sqsubset_p M$ and $\mathrm{St}(M, M) = \emptyset$.

(iii) By induction on $\mathrm{lh}(M)$. Let $L \sqsubset_p M \in V$. *Case* 1. $L \in \mathrm{Var}$. Then $\mathrm{St}(L, M) = \{M\}$ (if $M \notin \mathrm{Var}$) or $L \cong M$ (if $M \in \mathrm{Var}$). Hence $\|M\| \leq \|\mathrm{St}(L, M)\| + \|L\|$, since $L \cong M$ implies $\|L\| = \|M\|$, by (i). *Case* 2. $L \notin \mathrm{Var}$. Let $\|L\| = \|\mathrm{St}(K, L)\| + |K|$ with $K \sqsubset_p L$ and $L \notin \mathrm{St}(K, L)$. Then also $K \notin \mathrm{Var}$. We have: $K \sqsubset_p L \sqsubset_p M$. Hence, by a simple application of Lemma 2.10(i), $K \sqsubset_p M$. Moreover: $\forall N \in \mathrm{St}(K, M)\, \mathrm{lh}(N) < \mathrm{lh}(M)$ since $K \notin \mathrm{Var}$. Hence, by the induction hypothesis,

$$\forall N \in \mathrm{St}(K, M)\ \forall F \sqsubset_p N\ \|N\| \leq \|\mathrm{St}(F, N)\| + \|F\|.$$

From this it follows by a second application of lemma 2.10(i) that

$$\forall N \in \mathrm{St}(K, M)\ \|N\| \leq \|\mathrm{St}(L, M)\| + \|\mathrm{St}(K, L)\|.$$

Hence $\|\mathrm{St}(K, M)\| \leq \|\mathrm{St}(L, M)\| + \|\mathrm{St}(K, L)\|$ and we conclude:

$$\|M\| \leq \|\mathrm{St}(K, M)\| + |K| \leq \|\mathrm{St}(L, M)\| + \|\mathrm{St}(K, L)\| + |K| = \|\mathrm{St}(L, M)\| + \|L\|.$$

(iv) Let $|\cdot|$ be weakly reducing, i.e., $|\cdot|$ satisfies the conditions (Arg), (Fex) and (Suh) of 2.3. Now we must prove that $\|\cdot\|$ also satisfies these conditions.

As to (Arg). By induction $\mathrm{lh}(M)$ we show for each $M \in V$: $\forall M' \subset_{\mathrm{arg}} M\, \|M'\| < \|M\|$. Let $M' \subset_{\mathrm{arg}} M \in V$. Write $\|M\| = \|\mathrm{St}(L, M)\| + |L|$ with $L \sqsubset_p M$, $M \notin \mathrm{St}(L, M)$. Now apply Lemma 2.9(i): *Case* 1. For some argument L' of L, $L' \sqsubset M'$ and $\mathrm{St}(L', M') \subseteq \mathrm{St}(L, M)$. Then also $L' \sqsubset_p M'$ since $L \sqsubset_p M$. Hence $\|M'\| \leq \|\mathrm{St}(L', M')\| + |L'| \leq \|\mathrm{St}(L, M)\| + |L'|$. Moreover, since $L' \sqsubset_{\mathrm{arg}} L$ and $|\cdot|$ satisfies (Arg), $|L'| < |L|$. Therefore $\|M'\| < \|\mathrm{St}(L, M)\| + |L| = \|M\|$. *Case* 2. $M' \subset_{\mathrm{arg}} N$ for some $N \in \mathrm{St}(L, M)$. Then by the induction hypothesis:

$$\|M'\| < \|N\| \leqslant \|\mathrm{St}(L, M)\| \leqslant \|\mathrm{St}(L, M)\| + |L| = \|M\|.$$

As to (Fex). Let $M \in V(\sigma \to \tau)$ and $X \in \mathrm{Var}(\sigma)$. Again, write $\|M\| = \|\mathrm{St}(L, M)\| + |L|$ with $L \sqsubset_p M$. Then clearly $LX \sqsubset_p MX$ with $\mathrm{St}(LX, MX) = \mathrm{St}(L, M)$. Hence $\|MX\| \leqslant \|\mathrm{St}(LX, MX)\| + |LX| = \|\mathrm{St}(L, M)\| + |LX|$. Since $|\cdot|$ satisfies (Fex) we have: $|LX| \leqslant |L|$ and, moreover, $|LX| < |L|$ if $\tau = 0$. Hence $\|MX\| \leqslant \|\mathrm{St}(L, M)\| + |L| = \|M\|$ with, furthermore, $<$ instead of $\leqslant$ if $\tau = 0$.

As to (Suh). Let $M \in V$; we must prove: $\forall M' \leftarrow M\ \|M'\| < \|M\|$ or $\exists G \subset_h M\ \|G\| < \|M\|$. Let $\|M\| = \|\mathrm{St}(L, M)\| + |L|$ with $L \sqsubset_p M$, $M \notin \mathrm{St}(L, M)$. Since $|\cdot|$ satisfies (Suh) we have: (a) $\forall L' \leftarrow L\ |L'| < |L|$ or (b) for some $F \subset_h L\ |F| < |L|$. We split up (a) into: (a1) L is not critical (w.r.t. M), (a2) L is critical. For each of these cases (a1), (a2) and (b) we show that the condition $[(\forall M' \leftarrow M\ \|M'\| < \|M\|) \vee \exists G \subset_h M\ \|G\| < \|M\|]$ is satisfied indeed.

(a1) First we prove that in this case L cannot be an improper frame of M (cf. 2.9). Suppose that L is improper; then we have: $\mathrm{Typ}(L) = \mathrm{Typ}(M) = 0$ and $L \equiv BtK_1 \cdots K_k$, $M \equiv B\bar{n}L_1 \cdots L_k$ with B a recursion block, $n \in \mathbb{N}$, t not a numeral. Write $L' \equiv B\bar{n}K_1 \cdots K_k$. Then clearly, because of $L \sqsubset_p M$, $L' \sqsubset_p M$ and $\mathrm{St}(L', M) \subseteq \mathrm{St}(L, M)$. Hence

$$\|M\| \leqslant \|\mathrm{St}(L', M)\| + |L'| \leqslant \|\mathrm{St}(L, M)\| + |L'|.$$

But also $L \to L'$ (by clause (4) in 2.2) and consequently $|L'| < |L|$ (by the assumption (a)). So it follows that $\|M\| < \|\mathrm{St}(L, M)\| + |L| = \|M\|$, a contradiction.

After this observation we can apply Lemma 2.9(ii) as follows to the present case (a1). Let $M \to M'$. Then, since L is neither critical nor improper w.r.t. M, there is a successor L' of L such that $L' \sqsubset M'$ and $\mathrm{St}(L', M') \subseteq \mathrm{St}(L, M)$. Then also $L' \sqsubset_p M'$ (since $L \sqsubset_p M$) and, moreover, $|L'| < |L|$ by assumption (a). Hence

$$\|M'\| \leqslant \|\mathrm{St}(L', M')\| + |L'| < \|\mathrm{St}(L, M)\| + |L| = \|M\|.$$

(a2) Let $L \equiv XK_1 \cdots K_k$ and $M \equiv \mathrm{G}L_1 \cdots L_k$, where X is a variable and G is not a variable. Then, because of $L \sqsubset_p M$, $G \in \mathrm{St}(L, M)$ and $\mathrm{Lev}(G) \leqslant p$. From the assumption $M \notin \mathrm{St}(L, M)$ it follows that $k \geqslant 1$ and, consequently, $G \subset_h M$. Since $|\cdot|$ satisfies (Arg), $k \geqslant 1$ implies also $|L| > 0$ ($k \geqslant 1 \Rightarrow K_1 \subset_{\mathrm{arg}} L \Rightarrow |K_1| < |L|$). Hence, using $G \in \mathrm{St}(L, M)$ we can simply conclude $\|G\| < \|M\|$ by

$$\|G\| \leqslant \mathrm{St}(L, M)\| < \|\mathrm{St}(L, M)\| + |L| = \|M\|.$$

(b) Let $L \equiv FK_1 \cdots K_k$ with $k \geqslant 1$, $|F| < |L|$. Then we can write, because of $L \sqsubset_p M$: $M \equiv GL_1 \cdots L_k$ with $F \sqsubset_p G$, $\mathrm{St}(F, G) \subseteq \mathrm{St}(L, M)$ (and also $K_i \sqsubset_p L_i$, $\mathrm{St}(K_i, L_i) \subseteq \mathrm{St}(L, M)$ for $i = 1, \ldots, k$). It follows that $G \subset_h M$ (with $\mathrm{Lev}(G) = \mathrm{Lev}(F)$) and $\|G\| < \|M\|$ because of

$$\|G\| \leqslant \|\mathrm{St}(F, G)\| + |F| \leqslant \|\mathrm{St}(L, M)\| + |F| < \|\mathrm{St}(L, M)\| + |L| = \|M\|.$$

This completes the proof of the fact that $|\cdot|$ is also weakly reducing. The remaining part of (iv) follows immediately by inspection of the above argumentation. (In particular as to (2) in (iv): observe that in the treatment of case (a2) above we did not use the assumption (a) at all.)

(v) First we prove the following claim, using only the fact that $|\cdot|$ satisfied (Fex).

Claim. *Suppose*: $L \in V$, $\mathrm{Typ}(L) = 0$. *Then*

$$\forall F \subset_{\mathrm{h}} L\, [\mathrm{Lev}(F) \leqslant p+1 \Rightarrow \exists K \subset_{\mathrm{arg}} L\ \|L\| < |K| + |F|].$$

Proof. Let $L \equiv FK_1 \cdots K_k$ with $k \geqslant 1$, $\mathrm{Lev}(F) \leqslant p+1$. Choose variables $X_1, \ldots, X_k$ of the same types as $K_1, \ldots, K_k$ respectively. Then $FX_1 \cdots X_k \sqsubset L$ and $\mathrm{St}(F\vec{X}, L) = \{K_1, \ldots, K_k\} \setminus \mathrm{Var}$. Moreover: $F\vec{X} \sqsubset_{\mathrm{p}} L$, since for each i ($1 \leqslant i \leqslant k$): $\mathrm{Lev}(K_i) < \mathrm{Lev}(F) \leqslant p+1$ and, consequently, $\mathrm{Lev}(K_i) \leqslant p$.

Hence, by the definition of $\|L\|$,

$$\|L\| \leqslant \|\mathrm{St}(F\vec{X}, L)\| + |F\vec{X}| \leqslant \max\{\|K_1\|, \ldots, \|K_k\|\} + |F\vec{X}|.$$

Fix $K \in \{K_1, \ldots, K_k\}$ such that $\|K\| = \max\{\|K_1\|, \ldots, \|K_k\|\}$. Then $K \subset_{\mathrm{arg}} L$ and, since also $\|K\| \leqslant |K|$ (see (ii) above), $\|L\| \leqslant |K| + |F\vec{X}|$. Since $\mathrm{Typ}(F\vec{X}) = \mathrm{Typ}(L) = 0$ and $k \geqslant 1$ it follows also, by repeated application of (Fex), that $|F\vec{X}| < |F|$. Hence $\|L\| < |K| + |F|$, as desired. □ Claim

After this we proceed as follows. By assumption: $p < \omega$, $|\cdot|$ has rank $p+1$ and $\forall K, L \in V$ ($|K| < |L| \Rightarrow |K| + |K| \leqslant |L|$). Obviously this implies:

$$(*) \qquad \forall K, F, L \in V\, [(|K| < |L| \wedge |F| < |L|) \Rightarrow |K| + |F| \leqslant |L|]$$

By (iv) above we already know that the ordinal assignment $\|\cdot\|$ is of rank $p+1$ (take $q = p+1$). In order to prove that $\|\cdot\|$ is also of rank p we must show:

$$\forall M \in V[(\forall M' \leftarrow M\, \|M'\| < \|M\|) \vee \exists G \subset_{\mathrm{h}} M\, (\|G\| < \|M\| \wedge \mathrm{Lev}(G) \leqslant p)].$$

So let $M \in V$ be given. Write $\|M\| = \|\mathrm{St}(L, M)\| + |L|$ with $L \sqsubset_{\mathrm{p}} M$, $M \notin \mathrm{St}(L, M)$. Since $|\cdot|$ is of rank $p+1$ we have (a) $\forall L' \leftarrow L\ |L'| < |L|$ or (b) for some $F \subset_{\mathrm{h}} L$, $|F| < |L|$ and $\mathrm{Lev}(F) \leqslant p+1$. If (a) applies, then we are done because of (1) and (2) in (iv). We are also done if $\mathrm{Typ}(M) \neq 0$ (or equivalently: $\mathrm{Typ}(L) \neq 0$), for in that case each successor M' is of the form $MX_1 \cdots X_k$ with $X_1, \ldots, X_k \in \mathrm{Var}$ and $\mathrm{Typ}(M\vec{X}) = 0$, whence $\forall M' \leftarrow M\, \|M'\| < \|M\|$ since $\|\cdot\|$ satisfies (Fex) (see (iv)).

So it remains to consider the following case: $\mathrm{Typ}(L) = 0$ and (b) holds. But this case cannot apply. For suppose that $\mathrm{Typ}(L) = 0$ and F is as in (b). Then, by the above claim, $\|L\| < |K| + |F|$ for some $K \subset_{\mathrm{arg}} L$. Moreover: $|K| + |F| \leqslant |L|$ because of $(*)$. (Recall that $|K| < |L|$ holds since $|\cdot|$ satisfies (Arg) and that $|F| < |L|$ holds by assumption.) Hence $\|L\| < |L|$ and, consequently, $\|\mathrm{St}(L, M)\| + \|L\| < \|\mathrm{St}(L, M)\| + |L| = \|M\|$. But this contradicts (iii) above. □

2.12. Proof of Theorem 2.6. Let $|\cdot|: V \to \mathrm{ON}$ and $\|\cdot\| = \mathrm{RED}_p(|\cdot|): V \to \mathrm{ON}$ be as in the statement of Theorem 2.6; so $|\cdot|$ is of rank $p+1$, where $p<\omega$. Write $|\cdot|' = \exp(|\cdot|)$; then $\|\cdot\| = \mathrm{SUB}_p(|\cdot|')$. It is obvious that $|\cdot|'$ is of rank $p+1$ again; this follows from a glance at the definitions in 2.3, recalling that the operation $\exp: \mathrm{ON} \to \mathrm{ON}$ is strictly increasing. Moreover: the property $\exp(\alpha)+\exp(\alpha) = \exp(\alpha+1)$ of exp (combined with the monotonicity of exp) implies $\forall K, L \in V$ $(|K|' < |L|' \Rightarrow |K|' + |K|' \leqslant |L|')$. Hence, by Lemma 2.11(v) (applied to $|\cdot|'$ instead of $|\cdot|$), $\|\cdot\|$ is of rank p, indeed. And also: if $M \in V$, then by Lemma 2.11(ii), $\|M\| \leqslant |M|' = \exp(|M|)$. $\square$

2.13. In order to prove Lemma 2.8 we first list some more basic properties of the operation SUB_p $(0 \leqslant p \leqslant \omega)$, as defined in 2.5:

Lemma. *Let* $|\cdot|$, $|\cdot|': V \to \mathrm{ON}$ *be ordinal assignments and let* $0 \leqslant q \leqslant p \leqslant \omega$. *Then the following hold:*

(i) $\mathrm{SUB}_p(|\cdot|) \leqslant \mathrm{SUB}_q(|\cdot|)$ (*i.e.*, $\forall M \in V\, \mathrm{SUB}_p(|\cdot|)(M) \leqslant \mathrm{SUB}_q(|\cdot|)(M)$).

(ii) $\mathrm{SUB}_q(\mathrm{SUB}_p(|\cdot|)) = \mathrm{SUB}_p(|\cdot|)$.

(iii) $\forall M \in V\, [\mathrm{Lev}^*(M) \leqslant q \;\Rightarrow\; \mathrm{SUB}_p(|\cdot|)(M) = \mathrm{SUB}_q(|\cdot|)(M)]$.

(iv) $\forall M \in V\, [(\forall N \subset M\, \forall L \sqsubset N\, |L| \leqslant |L|') \;\Rightarrow\; \mathrm{SUB}_p(|\cdot|)(M) \leqslant \mathrm{SUB}_p(|\cdot|')(M)]$.

In consequence: $|\cdot| \leqslant |\cdot|' \;\Rightarrow\; \mathrm{SUB}_p(|\cdot|) \leqslant \mathrm{SUB}_p(|\cdot|')$.

Proof. The proofs of (i), (iii) and (iv) are straightforward, by induction on the length $\mathrm{lh}(M)$ of terms $M \in V$. (As to (i) and (iii): for all terms L and M we have because of $q \leqslant p$: $L \sqsubset_q M \Rightarrow L \sqsubset_p M$, respectively $\mathrm{Lev}^*(M) \leqslant q \Rightarrow (L \sqsubset_q M \Leftrightarrow L \sqsubset_p M)$. As to (iv): observe that, in general, $L \sqsubset N \subset K \sqsubset M \Rightarrow \exists N'\, L \sqsubset N' \subset M$; this enables us to make the induction step.)

(ii) Write $\|\cdot\| = \mathrm{SUB}_p(|\cdot|)$ and $|||\cdot||| = \mathrm{SUB}_q(\mathrm{SUB}_p(|\cdot|)) = \mathrm{SUB}_q(\|\cdot\|)$. We already know, by Lemma 2.11(ii), that $|||\cdot||| \leqslant \|\cdot\|$. We will now prove by induction on $\mathrm{lh}(M)$ that also $\|M\| \leqslant |||M|||$ for each $M \in V$; altogether this implies $|||\cdot||| = \|\cdot\|$. So let $M \in V$ and let $|||M||| = |||\mathrm{St}(L, M)||| + \|L\|$ with $L \sqsubset_q M$ and $M \notin \mathrm{St}(L, M)$. Then also $L \sqsubset_p M$ since $q \leqslant p$. Hence, by Lemma 2.11(iii), $\|M\| \leqslant \|\mathrm{St}(L, M)\| + \|L\|$. Moreover, by the induction hypothesis, $\|\mathrm{St}(L, M)\| \leqslant |||\mathrm{St}(L, M)|||$. Therefore $\|M\| \leqslant |||\mathrm{St}(L, M)||| + \|L\| = |||M|||$. $\square$

2.14. Proof of Lemma 2.8. Let $(V, h) = (\lambda^{\tau}, \mathrm{Lev}^*)$ or $(V, h) = (\mathrm{TR}, \mathrm{Deg})$. Let $|\cdot|: V \to \mathrm{ON}$ be an ordinal assignment. Write $|\cdot|_{n,p} = \mathrm{RED}_{n,p}(|\cdot|)$ $(p \leqslant n < \omega)$. In view of the definition of $\mathrm{RED}_*(|\cdot|)(M)$ (at the beginning of 2.8) it suffices to prove:

$$\forall n < \omega\, \forall M \in V\, [\mathrm{Lev}^*(M) + 1 \leqslant n \;\Rightarrow\; |M|_{n,0} \leqslant |M|_{n+1,0}].$$

So let $n<\omega$ be given, $n\geq 1$. By induction on $n-p$ we prove, more generally, that for each $p<n$:

$$\forall M\in V\,[\mathrm{Lev}^*(M)+1\leq n\ \Rightarrow\ |M|_{n,p}\leq |M|_{n+1,p}].$$

$p=n-1$: By definition, $|\cdot|_{n,n-1}=\mathrm{RED}_{n-1}(|\cdot|)$ and $|\cdot|_{n+1,n-1}=\mathrm{RED}_{n-1}(\mathrm{RED}_n(|\cdot|))$. Using Lemma 2.13 we derive:

(1) For all $M\in V$ with $\mathrm{Lev}^*(M)\leq n-1$:

$$\begin{aligned}|M|_{n,n-1}&=\mathrm{RED}_{n-1}(|\cdot|)(M)=\mathrm{SUB}_{n-1}(\exp(|\cdot|))(M)\\&=\mathrm{SUB}_n(\exp(|\cdot|))(M)=\mathrm{RED}_n(|\cdot|)(M)\qquad\text{(see 2.13(iii))}.\end{aligned}$$

(2)
$$\begin{aligned}\mathrm{RED}_n(|\cdot|)&=\mathrm{SUB}_n(\exp(|\cdot|))=\mathrm{SUB}_{n-1}(\mathrm{SUB}_n(\exp(|\cdot|))\\&=\mathrm{SUB}_{n-1}(\mathrm{RED}_n(|\cdot|))\leq \mathrm{SUB}_{n-1}(\exp(\mathrm{RED}_n(|\cdot|)))\\&=\mathrm{RED}_{n-1}(\mathrm{RED}_n(|\cdot|))=|\cdot|_{n+1,n-1}.\end{aligned}$$

Hence $|M|_{n,n-1}=\mathrm{RED}_n(|\cdot|)(M)\leq |M|_{n+1,n-1}$ for all $M\in V$ with $\mathrm{Lev}^*(M)+1\leq n$.

$p<n-1$: By definition, $|\cdot|_{n,p}=\mathrm{RED}_p(|\cdot|_{n,p+1})=\mathrm{SUB}_p(\exp(|\cdot|_{n,p+1}))$ and $|\cdot|_{n+1,p}=\mathrm{RED}_p(|\cdot|_{n+1,p+1})=\mathrm{SUB}_p(\exp(|\cdot|_{n+1,p+1}))$. The induction hypothesis implies: $\exp(|M|_{n,p+1})\leq\exp(|M|_{n+1,p+1})$ for all $M\in V$ with $\mathrm{Lev}^*(M)+1\leq n$. Now apply Lemma 2.13(iv). □

2.15. This completes the first part of this section, which concerned the reduction of the ranks of already given weakly reducing ordinal assignments. We now turn to the actual construction of these.

First we settle, for later reference, a lemma about the so-called property (Fra); by definition, an ordinal assignment $|\cdot|:V\to\mathrm{ON}$ *satisfies* (Fra) if

$$\forall M\in V\,\forall L\sqsubset M\ |L|\leq|M|.$$

Lemma. *Let $|\cdot|:V\to\mathrm{ON}$ be an ordinal assignment and let $0\leq p\leq\omega$. Suppose that $|\cdot|$ satisfies* (Fra). *Then $\mathrm{SUB}_p(|\cdot|)$ also satisfies* (Fra).

Proof. Let $\|\cdot\|=\mathrm{SUB}_p(|\cdot|)$. We prove by induction on $\mathrm{lh}(M)$ that, for each $M\in V$, $\forall L\sqsubset M\ \|L\|\leq\|M\|$. So let $L\sqsubset M\in V$. Write $\|M\|=\|\mathrm{St}(K,M)\|+|K|$ with $K\sqsubset_p M$, $M\notin\mathrm{St}(K,M)$. By Lemma 2.10(ii) there exists an infimum $\inf(K,L)$ of K and L w.r.t. $\sqsubset$. Moreover:

$$\forall N\in\mathrm{St}(\inf(K,L),L)\ \exists F\in\mathrm{St}(K,M)\ N\sqsubset F.$$

Hence $\inf(K,L)\sqsubset_p L$ (since also $K\sqsubset_p M$) and further, by the induction hypothesis (applied to $F\in\mathrm{ST}(K,M)$), $\|\mathrm{St}(\inf(K,L),L)\|\leq\|\mathrm{St}(K,M)\|$. Furthermore: $|\inf(K,L)|\leq|K|$, since $\inf(K,L)\sqsubset K$ and $|\cdot|$ satisfies (Fra). We conclude:

$$\|L\|\leq\|\mathrm{St}(\inf(K,L),L)\|+|\inf(K,L)|\leq\|\mathrm{St}(K,M)\|+|K|=\|M\|.\qquad\square$$

2.16. Construction IA. We construct as follows a map $|\cdot|:\lambda^\tau\to\omega$. First we define, by recursion on the length of terms $M\in\lambda^\tau$, an auxiliary map $f:\lambda^\tau\to\omega$.

1. If $M \equiv AK_1 \cdots K_k$ with $k \geqslant 0$ and $A \in \mathrm{Var} \cup \{\bar{0}, \bar{S}\}$, then

$$f(M) = \max\{\mathrm{Lev}(A), f(K_1)+1, \ldots, f(K_k)+1\}.$$

2. If $M \equiv (\lambda XG)K_1 \cdots K_k$ with $k \geqslant 0$, then

$$f(M) = \max\{\mathrm{Lev}(X), f(G), f(K_1), \ldots, f(K_k)\}+1.$$

Next we define for each $M \in \lambda^\tau$:

$$|M| = \begin{cases} 3f(M) & \text{if } \mathrm{Typ}(M)=0 \text{ and } \forall L \subset_{\mathrm{arg}} M\, L \in \mathrm{Var}, \\ 3f(M)+1 & \text{if } \mathrm{Typ}(M) \neq 0 \text{ and } \forall L \subset_{\mathrm{arg}} M\, L \in \mathrm{Var}, \\ 3f(M)+2 & \text{otherwise (i.e., if } \exists L \subset_{\mathrm{arg}} M\, L \notin \mathrm{Var}). \end{cases}$$

Lemma. *This map* $|\cdot| : \lambda^\tau \to \omega$ *is a weakly reducing ordinal assignment. Moreover: it satisfies* (Fra) (*cf.* 2.15).

Proof. It suffices to prove the following (cf. 2.2 and 2.3):
(1) $\forall M \in \lambda^\tau\, \forall L \cong M\, |L| = |M|$.
(2) $|\cdot|$ satisfies (Arg), (Fex) and (Fra).
(3) Let $M \equiv (\lambda XG)HK_1 \cdots K_k \in \lambda^\tau$, $\mathrm{Typ}(M)=0$. Then:
a. If $H, K_1, \ldots, K_k \in \mathrm{Var}$, then $|G[H/X]K_1 \cdots K_k| < |M|$.
b. If $\exists L \in \{H, K_1, \ldots, K_k\}\, L \notin \mathrm{Var}$, then $|\lambda XG| < |M|$.
But this is a routine exercise; the details are left to the reader. □

Construction IB. Let $|\cdot| : \lambda^\tau \to \omega$ be as constructed above. From $|\cdot|$ we construct the following map $|\cdot|_\lambda : \lambda^\tau \to \mathrm{ON}$ (cf. 2.8):

$$|\cdot|_\lambda = \mathrm{RED}_*(|\cdot|).$$

Corollary. $|\cdot|_\lambda$ *is a reducing ordinal assignment and for all* $M \in \lambda^\tau$ *we have*: $|M|_\lambda \leqslant 2_n(|M|) < \omega$, *where* $n = \mathrm{Lev}^*(M)$. *Moreover*: $|\cdot|_\lambda$ *satisfies* (Fra).

Proof. The first part follows immediately from the above Lemma and Theorem 2.8 (see also the remark after the latter).

As to the second part ("$|\cdot|_\lambda$ satisfies (Fra)"): after inspection of the construction of $|\cdot|_\lambda = \mathrm{RED}_*(|\cdot|)$ from $|\cdot|$ (see Definition 2.8) it is clear that his follows ultimately from the following two facts.

(1) $|\cdot|$ satisfies (Fra) (see the above lemma).

(2) In general: if $|\cdot|' : V \to \mathrm{ON}$ is any ordinal assignment that satisfies (Fra), then the same holds for $\exp(|\cdot|')$ (this is obvious) and, consequently, also for $\mathrm{RED}_p(|\cdot|') = \mathrm{SUB}_p(\exp(|\cdot|'))$, for all $p \leqslant \omega$ (see Lemma 2.15). □

2.17. Using the reducing ordinal assignment $|\cdot|_\lambda : \lambda^\tau \to \omega$, as in construction IB, we will now construct a map $|\cdot|_{\mathrm{R}} : \mathrm{TR} \to \mathrm{ON}$; afterwards we will prove that $|\cdot|_{\mathrm{R}}$ is a weakly reducing ordinal assignment and, moreover, that for each $n \in \mathbb{N}$ the restriction of $|\cdot|_R$ to TR_n is of rank n.

We will make use of the following technical notions:

(i) A *standard frame* of a term M is a frame L of M with the following properties: 1. each free variable of L has exactly one free occurrence in L and 2. if $\underset{\sim}{X}$ is a free occurrence of variable X in L and if $Y_1, \ldots, Y_k$ $(k \geq 0)$ are the variables that have a free occurrence in L at the left of $\underset{\sim}{X}$, then X is just the first variable of the proper type such that $X \notin \{Y_1, \ldots, Y_k\}$ and X is not bound in L.

It is clear that M has only finitely many standard frames and, furthermore, that for each frame L of M there exists a unique standard frame K of M such that $K \cong L$.

(ii) By a *recursion redex* we mean a term of the form B or $B\bar{n}$, where B is a recursion block and $n \in \mathbb{N}$.

(iii) The (*numeral free*) λ^τ-*frame* M^* of term M is the (unique) standard frame L of M with the following properties; here we write $M \equiv L[K_1, \ldots, K_k/X_1, \ldots, X_k]$ with $\{X_1, \ldots, X_k\} = \mathrm{FV}(L)$ and, consequently, $\mathrm{St}(L, M) = \{K_1, \ldots, K_k\} \backslash \mathrm{Var}$. 1. $L \in \lambda^\tau$ and L contains no numerals. 2. Each element of $\mathrm{St}(L, M)$ is a numeral or a recursion redex. 3. If $1 \leq i \leq k$ and K_i is a numeral, then the free occurrence of X_i in L is neither in the context $\bar{S}X_i$ nor in a context of the form X_jX_i with $1 \leq j \leq k$ and K_j a recursion block.

In addition we define: $\mathrm{Rec}(M)$ is the set of all recursion redices that belong to $\mathrm{St}(M^*, M)$ (with M^* the λ^τ-frame of M, as above).

Remark. The elements of $\mathrm{Rec}(M)$ are just the recursion redices that have a maximal occurrence in M.

(iv) *A recursion term* is a term of the form $BK_1 \cdots K_k$ with B a recursion block, $k \geq 0$. The *head* (*recursion*) *redex* of such a recursion term $BK_1 \cdots K_k$ (with B a recursion block) is BK_1 if $k \geq 1$ and K_1 is a numeral and it is B otherwise. (So the head redex of a recursion term belongs to $\mathrm{Rec}(M)$.)

(v) $L \subset_{\mathrm{rarg}} M$ means that L is a *recursion argument* of M; that is: M is a recursion term and if we write $M \equiv FK_1 \cdots K_k$ with F the head redex of M, then $L \in \{K_1, \ldots, K_k\}$.

(vi) If M is any term of type $\sigma_1 \to \cdots \to \sigma_k \to 0$ with $k \geq 1$, then

$$M^+ \equiv MX_1 \cdots X_k$$

where, for each i $(1 \leq i \leq k)$, X_i is the first variable of type σ_i. (So if $\mathrm{Typ}(M) \neq 0$, then $M \to M^+$ by clause (1) in 2.2.)

2.18. Construction IIA. In order to construct the map $|\cdot|_R : \mathrm{TR} \to \mathrm{ON}$ we first construct two auxiliary maps $g, f : \mathrm{TR} \to \mathrm{ON}$. g is defined by recursion on the length of terms: (recall that M^* is the λ^τ-frame of term M, as defined in 2.17(iii) above)

1. If M is not a recursion redex (cf. 2.17(ii)), then

$$g(M) = \max\{g(B) \mid B \in \mathrm{Rec}(M)\} \quad (=0 \text{ if } M \in \lambda^\tau).$$

2. $g(RGH) = \max(g(G), g(H) + \omega) + \omega$.
3. $g(RGH\bar{n}) = \max(g(G), g(H) + \omega) + |G^*|_\lambda \cdot n$.
4. $g(TG) = g(G) + |{<}| + \omega$.
5. $g(TG\bar{n}) = g(G) + (|G^*|_\lambda + 2)(|n| + 1)$.
6. $g([T]G\bar{n}) = g(G) + (|G^*|_\lambda + 2) \cdot |n| + 2$.

7a. $g([T]G\bar{n}\bar{m}) = g(G) + (|G^*|_\lambda + 2)(|m| + 1) + 1$ if $m < n$.

7b. $g([T]G\bar{n}\bar{m}) = 0$ if $\neg m < n$.

Next f is defined by:

$$f(M) = \begin{cases} |M|_\lambda & \text{if } M \in \lambda^\tau, \\ g(M) + |M^*|_\lambda & \text{if } M \notin \lambda^\tau. \end{cases}$$

After these preparations we finally define:

$$|M|_{\mathrm{R}} = \begin{cases} f(M) & \text{if } M \in \lambda^\tau, \\ \omega + 3f(M^+) + 1 & \text{if } M \text{ is a recursion term and } \mathrm{Typ}(M) \neq 0 \text{ and} \\ & \quad \forall L \subset_{\mathrm{rarg}} M\, L \in \mathrm{Var}, \\ \omega + 3f(M) + 2 & \text{if } \exists L \subset_{\mathrm{rarg}} M\, L \notin \mathrm{Var}, \\ \omega + 3f(M) & \text{otherwise} \end{cases}$$

(where M^+ is as in 2.17(vi)).

Construction IIB. $\|\cdot\|_{\mathrm{R}} = \mathrm{RED}_*(|\cdot|_{\mathrm{R}})$.

2.19. Theorem. (i) *The map* $|\cdot|_{\mathrm{R}} : \mathrm{TR} \to \mathrm{ON}$, *as in Construction* IIA, *is a weakly reducing ordinal assignment. For each* $n \in \mathbb{N}$ *the restriction of* $|\cdot|_{\mathrm{R}}$ *to* TR_n *is of rank* n. *Moreover: the ordinals assigned by* $|\cdot|_{\mathrm{R}}$ *are bounded as follows:*

$$\forall M \in \mathrm{TR}\; |M|_{\mathrm{R}} < |{<}| \cdot \omega.$$

(ii) *The map* $\|\cdot\|_{\mathrm{R}} : \mathrm{TR} \to \mathrm{ON}$, *as in Construction* IIB, *is a reducing ordinal assignment* (*in other words: an ordinal assignment of rank* 0). *Moreover: for all terms* $M \in \mathrm{TR}$,

$$\|M\|_{\mathrm{R}} < 2_n(|{<}| \cdot \omega) \quad \text{with } n = \mathrm{Deg}(M).$$

In consequence, the successor relation $\to$ *is wellfounded and the tree that is generated from any term by repeated application of the successor relation has length less than the first* ε*-number* $> |{<}| \cdot \omega$.

Proof. We start with listing a number of immediate consequences C1, C2, ..., C10 of the definitions in Construction IIA. We abbreviate:

$$\mathrm{RV} = \{M \in \mathrm{TR} \mid M \text{ is a recursion term}, \; \mathrm{Typ}(M) \neq 0 \text{ and } \forall L \subset_{\mathrm{rarg}} M\, L \in \mathrm{Var}\}$$

(with R and V referring to 'recursion' and 'variable', respectively). So if $M \in \mathrm{RV}$, then $|M|_{\mathrm{R}} = \omega + 3f(M^+) + 1$.

C1. g, f and $|\cdot|_{\mathrm{R}}$ are ordinal assignments again (in the sense of 2.3).

C2. $\forall M \in \mathrm{TR}\ g(M) = \max\{g(B) \mid B \in \mathrm{Rec}(M)\}$. So $\mathrm{Rec}(L) \subseteq \mathrm{Rec}(M) \Rightarrow g(L) \leqslant g(M)$.

C3. f satisfies (Fex) (since $|\cdot|_\lambda$ satisfies (Fex)).

C4. If $L \in \lambda^\tau$ and $M \notin \lambda^\tau$, then $|L|_{\mathrm{R}} = |L|_\lambda < \omega \leqslant |M|_{\mathrm{R}}$.

C5. If $L \notin \lambda^\tau$, $M \notin \lambda^\tau$ and $M \notin \mathrm{RV}$, then $f(L) < f(M) \Rightarrow |L|_{\mathrm{R}} < |M|_{\mathrm{R}}$. (If $L \in \mathrm{RV}$, then observe that because of C3: $f(L^+) < f(L)$ and consequently $|L|_{\mathrm{R}} = \omega + 3f(L^+) + 1 < \omega + 3f(L)$.)

C6. If B is a recursion block and $n \in \mathbb{N}$, then $g(B\bar{n}) < g(B)$. (This is obvious if $B \equiv \mathrm{RGH}$. If $B \equiv TG$, then make use of: $\forall \alpha \in \mathrm{On}\, \forall p < \omega\ \ p\alpha < \alpha + \omega$. If $B \equiv [T]G\bar{m}$, with $n \prec m$ and consequently $|n| + 1 \leqslant |m|$, then we have:

$$g(B\bar{n}) = g(G) + (|G^*|_\lambda + 2)(|n| + 1) + 1 \leqslant g(G) + (|G^*|_\lambda + 2) \cdot |m| + 1$$
$$< g(G) + (|G^*|_\lambda + 2) \cdot |m| + 2 = g(B).$$

Finally, if $B \equiv [T]G\bar{m}$ and $\neg n \prec m$, then $g(B\bar{n}) = 0 < 2 \leqslant g(B)$.)

C7. $f(H) < g(H) + \omega \leqslant g(RGH\bar{O})$ and $g(RGH\overline{n+1}) = g(RGH\bar{n}) + |G^*|_\lambda$.

C8. $g(G) \leqslant g(RGH\bar{n})$.

C9. $g(G) < g([T]G\bar{n})$ and $g(TG\bar{n}) = g([T]G\bar{n}) + |G^*|_\lambda$.

C10. $g([T]G\bar{n}\bar{m}) = g(TG\bar{m}) + 1$ if $m \prec n$.

Now we turn to the actual proof of the theorem. (ii) follows immediately from (i) in view of Theorem 2.8. So we restrict our attention to (i). To begin with: the bound $|\prec| \cdot \omega$ is clear by a simple inspection of the definitions in Construction IIA. (Recall that the order type $|\prec|$ of $\prec$ is infinite.)

By C1 the map $|\cdot|_{\mathrm{R}}$ is an ordinal assignment indeed. Now it is easy to see that it suffices to prove the following:

(1) $|\cdot|_{\mathrm{R}}$ satisfies (Fex).

(2) $|\cdot|_{\mathrm{R}}$ satisfies (Arg).

(3) Let $M \equiv (\lambda XG)HK_1 \cdots K_k$, $\mathrm{Typ}(M) = 0$. Then $|G[H/X]K_1 \cdots K_k|_{\mathrm{R}} < |M|_{\mathrm{R}}$.

(4) Let M be a recursion term of type 0 and let B be the head redex of M. Then:

a. If $\forall K \subset_{\mathrm{rarg}} M\ K \in \mathrm{Var}$, then $\forall L \leftarrow M\ |L|_{\mathrm{R}} < |M|_{\mathrm{R}}$.

b. If $\exists K \subset_{\mathrm{rarg}} M\ K \notin \mathrm{Var}$, then $|B|_{\mathrm{R}} < |M|_{\mathrm{R}}$.

As to (1). Let $M \in \mathrm{TR}(\sigma \to \tau)$, $X \in \mathrm{Var}(\sigma)$. We must prove: $|MX|_{\mathrm{R}} \leqslant |M|_{\mathrm{R}}$ and $(\tau = 0 \Rightarrow |MX|_{\mathrm{R}} < |M|_{\mathrm{R}})$. *Case* 1: $M \notin \mathrm{RV}$. Apply C3 and C5. *Case* 2: $M \in RV$. If $\tau \neq 0$, then we have: $MX \in RV$, $M^+ \cong (MX)^+$ and, consequently,

$$|MX|_{\mathrm{R}} = \omega + 3f((MX)^+) + 1 = \omega + 3f(M^+) + 1 = |M|_{\mathrm{R}}.$$

If $\tau = 0$, then $M^+ \cong MX$ and

$$|MX|_{\mathrm{R}} = \omega + 3f(MX) = \omega + 3f(M^+) < \omega + 3f(M^+) + 1 = |M|_{\mathrm{R}}.$$

As to (2). Let $L \subset_{\text{arg}} M$. *Case* 1: $M \in \lambda^\tau$. Then also $L \in \lambda^\tau$ and $|L|_R = |L|_\lambda < |M|_\lambda = |M|_R$ since $|\cdot|_\lambda$ satisfies (Arg). *Case* 2: $M \notin \lambda^\tau$ and $L \in \lambda^\tau$. Then $|L|_R < |M|_R$ by C4. *Case* 3: $M \notin \lambda^\tau$ and $L \notin \lambda^\tau$. Then $M \notin \text{RV}$, since if $M \in \text{RV}$, then $\forall K \subset_{\text{arg}} M$ $K \in \lambda^\tau$. It is easy to see that $\text{Rec}(L) \subseteq \text{Rec}(M)$ and $L^* \subset_{\text{arg}} M^*$ (modulo $\cong$). Hence $g(L) \leqslant g(M)$ and $|L^*|_\lambda < |M^*|_\lambda$ since $|\cdot|_\lambda$ satisfies (Arg). Therefore $f(L) < f(M)$ and also, by C5, $|L|_R < |M|_R$.

As to (3). Write $L \equiv G[H/X]K_1 \cdots K_k$. *Case* 1: $M \in \lambda^\tau$. Then also $L \in \lambda^\tau$ and $|L|_R = |L|_\lambda < |M|_\lambda = |M|_R$ since $|\cdot|_\lambda$ satisfies (Suc). *Case* 2: $M \notin \lambda^\tau$ and $L \in \lambda^\tau$. Then $|L|_R < |M|_R$ by C4. *Case* 3: $M \notin \lambda^\tau$ and $L \notin \lambda^\tau$. Since $M \equiv (\lambda XG)HK_1 \cdots K_k$ the λ^τ-frame of M is of the form $M^* \equiv (\lambda XG')H'K_1' \cdots K_k'$, where $G', H', K_1', \ldots, K_k' \in \lambda^\tau$. Let $L' \equiv G'[H'/X]K_1' \cdots K_k'$. Then $|L'|_\lambda < |M^*|_\lambda$ since $|\cdot|_\lambda$ satisfies (Suc). On the other hand it follows by a simple inspection that $L^* \sqsubseteq L'$ and that each recursion redex $F \in \text{Rec}(L)$ belongs to $\text{Rec}(M)$ or is of the form $F_0\bar{n}$ with $F_0 \in \text{Rec}(M)$, $n \in \mathbb{N}$. Hence $|L^*|_\lambda \leqslant |L'|_\lambda$, since $|\cdot|_\lambda$ satisfies (Fra), and $g(L) \leqslant g(M)$ because of C2 and C6.

So it follows that $f(L) = g(L) + |L^*|_\lambda \leqslant g(M) + |L'|_\lambda < g(M) + |M^*|_\lambda = f(M)$ and, consequently, $|L|_R < |M|_R$ by C5. ($M \notin \text{RV}$ since $\text{Typ}(M) = 0$.)

As to (4a). Let $M \equiv BX_1 \cdots X_k$, $\text{Typ}(M) = 0$, where B is a recursion redex and X_i is a variable for each i $(1 \leqslant i \leqslant k)$. Suppose $M \to L$. We will prove that $|L|_R < |M|_R$. Choose $Y \in \text{Var}$ with $\text{Typ}(Y) = \text{Typ}(B)$. Then $\text{Rec}(M) = \{B\}$, $M^* \cong YX_1 \cdots X_k$ and $f(M) = g(B) + |YX_1 \cdots X_k|_\lambda$.

Case 1: B is a recursion block. Then $L \equiv X_1$ or $L \equiv B\bar{n}X_2 \cdots X_k$ for some $n \in \mathbb{N}$. If $L \equiv X_1$, then $|L|_R < |M|_R$ by C4. If $L \equiv B\bar{n}X_2 \cdots X_k$, then, for some $Z \in \text{Var}$, $L^* \cong ZX_2 \cdots X_k$, $\text{Rec}(L) = \{B\bar{n}\}$, $f(L) = g(B\bar{n}) + |ZX_2 \cdots X_k|_\lambda < g(B) + |YX_1 \cdots X_k|_\lambda = f(M)$, making use of C6 and the fact that $|\cdot|_\lambda$ satisfies (Fra). Hence $|L|_R < |M|_R$ by C5.

Case 2: $B \equiv RGH\bar{0}$. Then $L \equiv HX_1 \cdots X_k$. If $L \in \lambda^\tau$, then $|L|_R < |M|_R$ by C4. If $L \notin \lambda^\tau$, then, by C3 and C7, $f(L) \leqslant f(H) < g(B) \leqslant f(M)$ and, consequently, $|L|_R < |M|_R$ by C5.

Case 3: $B \equiv RGH\overline{n+1}$. Then $L \equiv G(RGH\bar{n})\bar{n}X_1 \cdots X_k$ and $\text{Rec}(L) = \text{Rec}(G) \cup \{RGH\bar{n}\}$, $L^* \cong G^*YxX_1 \cdots X_k$ with $x \in \text{Var}(0)$. Hence $g(L) = g(RGH\bar{n})$, by C2 and C8, and $|L^*|_\lambda < |G^*|_\lambda$ since $|\cdot|_\lambda$ satisfies (Fex). It follows that $f(L) = g(L) + |L^*|_\lambda < g(RGH\bar{n}) + |G^*|_\lambda = g(B) \leqslant f(M)$ (see C7) and therefore $|L|_R < |M|_R$ by C5.

Case 4: $B \equiv TG\bar{n}$. Then $L \equiv G([T]G\bar{n})\bar{n}X_1 \cdots X_k$, $\text{Rec}(L) = \text{Rec}(G) \cup \{[T]G\bar{n}\}$, $L^* \cong G^*YxX_1 \cdots X_k$. Hence $g(L) = g([T]G\bar{n})$ (see C9) and $|L^*|_\lambda < |G^*|_\lambda$. Therefore, by C9, $f(L) = g(L) + |L^*|_\lambda < g([T]G\bar{n}) + |G^*|_\lambda = g(B) \leqslant f(M)$ and, consequently, $|L|_R < |M|_R$ by C5.

Case 5: $B \equiv [T]G\bar{n}\bar{m}$. If $\neg m < n$, then $L \equiv \bar{0}$ and $|L|_R < |M|_R$ by C4. Now suppose $m < n$. Then $L \equiv TG\bar{m}X_1 \cdots X_k$ and, by C10,

$$
\begin{aligned}
f(L) &= g(L) + |L^*|_\lambda = g(TG\bar{m}) + |YX_1 \cdots X_k|_\lambda \\
&< g(TG\bar{m}) + 1 + |YX_1 \cdots X_k|_\lambda = g(B) + |YX_1 \cdots X_k|_\lambda = f(M).
\end{aligned}
$$

Hence $|L|_R < |M|_R$ by C5.

As to (4b): Let M be the recursion term $BK_1 \cdots K_k$, of type 0, with head redex B, where K_i is not a variable for some i $(1 \leq i \leq k)$. Write $M^* = YL_1 \cdots L_k$, $B^+ \equiv BX_1 \cdots X_k$. Then $|(B^+)^*|_\lambda = |YX_1 \cdots X_k|_\lambda \leq |YL_1 \cdots L_k|_\lambda = |M^*|_\lambda$ (since $|\cdot|_\lambda$ satisfies (Fra)). Also $g(B^+) = g(B) \leq g(M)$, since $B \in \mathrm{Rec}(M)$. Hence $f(B^+) \leq f(M)$. Now by the definition of $|\cdot|_R$ from f:

$$|B|_R = \omega + 3f(B^+) + 1 \leq \omega + 3f(M) + 1 < \omega + 3f(M) + 2 = |M|_R. \qquad \square$$

3. Reduction of higher type levels by means of transfinite recursion

Using the results of the preceding section we will now present the intended alternative proof fo Schwichtenberg's theorem about definitions of ordinal recursive functionals (see the introduction). We start with some preparations for a precise formulation of this theorem; these preparations include (1) the description of some standard constructions of w.o.'s from given w.o.'s and (2) the definition of the natural class of the so-called *good w.o.'s*, to which the theorem will apply. Together with (1) and (2) we will also state some simple corresponding facts that will be needed lateron.

3.1. We assume some standard coding $\langle\ \rangle$ of finite sequences of numbers to be given (for convenience with code 0 for the empty sequence), together with (primitive recursive) functions lh ('length'), π ('projection') and $*$ ('concatenation') such that

$$\mathrm{lh}(\langle x_0, \ldots, x_{n-1}\rangle) = n,$$

$$\pi(i, \langle x_0, \ldots, x_{n-1}\rangle) = x_i \quad \text{for } i < n,$$

$$\langle x_0, \ldots, x_{m-1}\rangle * \langle y_0, \ldots, y_{n-1}\rangle = \langle x_0, \ldots, x_{m-1}, y_0, \ldots, y_{n-1}\rangle.$$

Seq is the (primitive recursive) set of the sequence numbers.

If $x, y \in \mathrm{Seq}$, then $x \sqsubseteq y$ means: $\exists z \in \mathrm{Seq}\ x * z = y$ ('x is a prefix of y') and $x \sqsubset y : x \sqsubseteq y \wedge x \neq y$ ('x is a proper prefix of y').

Now let $\prec$ be any w.o. and let $k \leq \omega$. Then $\prec \cdot k$ is the following w.o. $\prec'$ of order type $|\prec| \cdot k$:

$$\mathrm{Field}(\prec') = \{\langle x, y\rangle \mid x \in \mathrm{Field}(\prec) \wedge y < k\}$$

and, for all $\langle x, y\rangle, \langle x', y'\rangle \in \mathrm{Field}(\prec')$,

$$\langle x, y\rangle \prec' \langle x', y'\rangle \Leftrightarrow y < y' \vee (y = y' \wedge x \prec x').$$

Further: $2^\prec$ is the following w.o. $\prec^*$ of order type $2^{|\prec|}$:

$$\mathrm{Field}(\prec^*) = \{x \in \mathrm{Seq} \mid \forall i < \mathrm{lh}(x)\ \pi(i, x) \in \mathrm{Field}(\prec)$$
$$\wedge \forall i\ (i + 1 < \mathrm{lh}(x) \Rightarrow \pi(i + 1, x) \prec \pi(i, x))\}$$

and, for all $x, y \in \mathrm{Field}(<^*)$,

$$x <^* y \iff x \sqsubset y \vee \exists z \in \mathrm{Seq}\, \exists a, b \in \mathbb{N}\, (z * \langle a \rangle \sqsubseteq x \wedge z * \langle b \rangle \sqsubseteq y \wedge a < b).$$

If $n \in \mathbb{N}$, then the *initial segment* of $<$ *up to* n is the w.o. $<\restriction n$ with

$$\mathrm{Field}(<\restriction n) = \{x \in \mathrm{Field}(<) \mid x < n\} \quad \text{and} \quad x <\restriction n\, y \iff x < y < n.$$

(So $|<\restriction n| = |n|_<$; cf. 1.2.) A w.o. $<'$ is called an *initial segment* of $<$ if $<' = <$ or $<' = <\restriction n$ for some $n \in \mathbb{N}$.

A *split embedding* of the w.o. $<$ into a w.o. $<'$ is a pair (f, g) of functions $f, g: \mathbb{N} \to \mathbb{N}$ such that

$$\forall x, y \in \mathbb{N}\, [(x < y \Rightarrow f(x) <' f(y)) \wedge (x \notin \mathrm{Field}(<) \Rightarrow f(x) = 0) \wedge (x \in \mathrm{Field}(<) \Rightarrow g(f(x)) = x)].$$

Lemma. *Let $<$ and $<'$ be w.o.'s and let $1 \leqslant k \leqslant \omega$. Then:*

(i) *If there exists a split embedding (f, g) of $<$ into $<'$ such that $f, g \in \mathrm{REC}_1(<')$, then for all $n \in \mathbb{N}: \mathrm{REC}_n(<) \subseteq \mathrm{REC}_n(<')$.*

(ii) *For all $n \in \mathbb{N}: \mathrm{REC}_n(<) \subseteq \mathrm{REC}_n(< \cdot k)$ and $\mathrm{REC}_n(<) \subseteq \mathrm{REC}_n(2^<)$.*

(iii) *The characteristic functions of $<, < \cdot k$ and $2^{< \cdot k}$ belong to $\mathrm{REC}_1(<)$.*

(iv) *If $0 \leqslant p \leqslant q \leqslant \omega$, then $< \cdot p$ is an initial segment of $< \cdot q$.*

(v) *If $<$ is an initial segment of $<'$, then $2^<$ is also an initial segment of $2^{<'}$.*

Proof. (i) By using a similar idea as in [7, §1.6].

(ii) By application of (i) with $<' = < \cdot k$, respectively $<' = 2^<$; in both cases the construction of appropriate functions f and g is easy. (As to $f \in \mathrm{REC}_1(<')$: make use of the fact that the characteristic function of $<'$ belongs to $\mathrm{REC}_1(<')$.)

(iii) By Lemma 1.2 we already know that $K_< \in \mathrm{REC}_1(<)$. The rest of (iii) is an easy consequence of this.

(iv) and (v) are obvious. □

3.2. Let $<$ be a w.o. We define as follows the so-called *basic operations* $p_<: \mathbb{N} \to \mathbb{N}$ ('predecessor') and $\oplus_<: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ ('addition modulo $<$') *of* $<$. For convenience we omit the index $<$ from $p_<, \oplus_<$ and $|x|_<$ $(x \in \mathbb{N})$.

If $|x|$ is a successor ordinal, then $p(x)$ is determined by: $p(x) \in \mathrm{Field}(<)$ and $|x| = |p(x)| + 1$. In all other cases $p(x) = 0$.

If $x, y \in \mathrm{Field}(<)$, then $x \oplus y$ is determined by: $x \oplus y \in \mathrm{Field}(<)$ and either $|x| + |y| = |x \oplus y|$ or $|x| + |y| = |<| + |x \oplus y|$. If $x \notin \mathrm{Field}(<)$ or $y \notin \mathrm{Field}(<)$, then $x \oplus y = 0$.

We call $<$ a *good w.o.* if $<$ is infinite (i.e. $\omega \leqslant |<|$) and these basic operations of $<$ both belong to $\mathrm{REC}_1(<)$. Note that this is a quite natural property; it applies, e.g., to the infinite standard wellorderings of order type $<\varepsilon_0$ as constructed in the usual way. (In fact, for these the functions $p_<$ and $\oplus_<$ are even primitive recursive.)

Lemma. *Let $\prec$ be a good w.o. and let $1 \leq k \leq \omega$. Then the basic operations of $\prec \cdot k$ and $2^{\prec \cdot k}$ belong to* $\mathrm{REC}_1(\prec)$. *In consequence, by Lemma* 3.1(ii), $\prec \cdot k$ *and* $2^{\prec \cdot k}$ *are good w.o.'s again.*

Proof. Easy exercise. (As to $2^{\prec \cdot k}$: make use of a similar argumentation as in [9, §2]; obviously the class of the 'good w.o.'s' is a subclass of the class of the 'nice w.o.'s' in the sense of [9, §2].) □

We are now able to present a precise formulation of the mentioned result of Schwichtenberg. (In fact, the condition on the w.o. $\prec$ below is not the literal version of Schwichtenberg's condition in [7, §3.8], but this difference does not seem to be essential.)

3.3. Theorem (Schwichtenberg [6], [7]). *Let $\prec$ be a good w.o. Suppose that $\Phi \in \mathrm{REC}_{n+1}(\prec)$ and that Φ has type level $\leq n$. Then for some $k \in \mathbb{N}$:*

$$\Phi \in \mathrm{REC}_n(2^{\prec \cdot k}).$$

3.4. The intended alternative proof of this theorem will take up the rest of this section from 3.5. Before beginning with it we state the following corollary, which is obtained by combination with the result of Terlouw [9] in the other direction. (Historically this result in the other direction goes back to earlier papers of Kreisel and Tait, but it seems that [9] is the only reference for a detailed proof of the general version that is incorporated in the Corollary here below. For a further discussion see the introduction of [9].)

Definition. Let $\prec$ be a w.o. and let $n, p \in \mathbb{N}$. Then

$$\mathrm{REC}_n(\prec) \restriction p = \{\Phi \in \mathrm{REC}_n(\prec) \mid \Phi \text{ has type level } \leq p\}.$$

Corollary. *Let $\prec$ be a good w.o. and let $n \in \mathbb{N}$. Then*

$$\mathrm{REC}_{n+1}(\prec) \restriction n = \bigcup_{k \in \mathbb{N}} \mathrm{REC}_n(2^{\prec \cdot k}) \restriction n.$$

3.5. *Conventions applying to the rest of this section.* From now on $\prec$ is a fixed good w.o., as in the statement of Theorem 3.3. n is a fixed number ≥ 1. (Observe that the theorem is trivially true for $n = 0$.) We assume that the successor relation $\rightarrow$ of the preceding section has been defined relative to this w.o. $\prec$ (see clauses (7a) and (7b) in 2.2.).

If M is a term and ρ is an assignment for M, then we mean by the value $[\![M]\!]_\rho$ of M the value $[\![M]\!]_\rho^\prec$ of M with respect to $\prec$ (see 1.4).

We assume that there is given a canonic coding

$$\ulcorner\ \urcorner : \mathrm{TR} \rightarrow \mathbb{N}: \quad M \mapsto \ulcorner M \urcorner$$

of terms into natural numbers such that the common syntactical operations on terms (like, e.g., substitution) as well as some other simple operations that will be used below correspond to primitive recursive functions; the construction of such a coding is routine.

3.6. Definition. (i) For convenience we denote the w.o. $2^{<\cdot\omega}$, of order type $\exp(|<|\cdot\omega)$, by $<^*$. We write $|x|=|x|_<$ and $|x|^*=|x|_{<^*}$ for $x\in\mathbb{N}$ (cf. 1.2).

(ii) Let $|\cdot|_R:\mathrm{TR}\to|<|\cdot\omega$ be the ordinal assignment as in 2.12, Construction IIA, and let $\mathrm{RED}_n(|\cdot|_R):\mathrm{TR}\to\exp(|<|\cdot\omega)$ be constructed from $|\cdot|_R$ as in 2.5. For convenience we will write from now on:

$$\|\cdot\|=\mathrm{RED}_n(|\cdot|_R).$$

Corresponding to this ordinal assignment $\|\cdot\|:\mathrm{TR}\to|<^*|$ we define as follows a function $\mathrm{ord}:\mathbb{N}\to\mathbb{N}$. If $M\in\mathrm{TR}$, then

$$\mathrm{ord}(\ulcorner M\urcorner)\in\mathrm{Field}(<^*)\quad\text{and}\quad|\mathrm{ord}(\ulcorner M\urcorner)|^*=\|M\|.$$

If $c\in\mathbb{N}$ and c is not the code of a term, then $\mathrm{ord}(c)=0$.

(iii) We define as follows the so-called *modified successor relation* $>$ between terms. (Instead of $M>L$ we will also write $L<M$.) Let term M be given. We consider two cases:

(H) $\exists F\subset_h M\,(\|F\|<\|M\|\wedge\mathrm{Lev}(F)\leqslant n)$. Then let F be the head term of M with minimal length such that $\|F\|<\|M\|$ and $\mathrm{Lev}(F)\leqslant n$. Write $M\equiv FK_1\cdots K_k$. Then by definition:

$$M>L\;\Leftrightarrow\;L\in\{F,K_1,\ldots,K_k\}.$$

(S) (H) does not apply to M. Then

$$M>L\;\Leftrightarrow\;M\to L.$$

In order to be able to formulate (in Lemma 3.8 below) some crucial properties of this modified successor relation $>$ and of the ordinal assignment $\|\cdot\|$ we need a few additional syntactical notions:

3.7. Definition. (i) If $\sigma,\tau\in\mathrm{Typ}$, then $\sigma\subset\tau$ means that σ is a *subtype* of τ; recursively defined:

$$\sigma\subset 0\;\Leftrightarrow\;\sigma=0\quad\text{and}\quad\sigma\subset\tau_1\to\tau_2\;\Leftrightarrow\;\sigma=\tau_1\to\tau_2\vee\sigma\subset\tau_1\vee\sigma\subset\tau_2.$$

If M is any term, then

$$\mathrm{Typ}^*(M)=\{\sigma\in\mathrm{Typ}\mid\exists L\subset M\,\sigma\subset\mathrm{Typ}(L)\}$$

(ii) By $\hat{M}$ we denote the *closure* of term M; that is:

$$\hat{M}\equiv\lambda X_1\cdots X_k\cdot M,$$

where $\{X_1,\ldots,X_k\}=FV(M)$ and $\ulcorner X_1\urcorner<\cdots<\ulcorner X_k\urcorner$. (If M is closed, then $k=0$ and $\hat{M}\equiv M$.)

3.8. Lemma. (i) $\forall M \in \mathrm{TR}_{n+1}\, \forall L < M\, \|L\| < \|M\| \wedge \mathrm{ord}(\ulcorner L \urcorner) \prec^* \mathrm{ord}(\ulcorner M \urcorner)$.

(ii) $\forall M \in \mathrm{TR}_{n+1}\, \forall L < M\, \mathrm{Typ}^*(L) \subseteq \mathrm{Typ}^*(M) \wedge \mathrm{Lev}(\hat{L}) \leqslant \max(n, \mathrm{Lev}(\hat{M}))$.

(iii) $\mathrm{ord} \in \mathrm{REC}_1(\prec)$.

Proof. (i) Let $|\cdot|'$ be the restriction $|\cdot|_{\mathrm{R}}$ to TR_{n+1}; then $|\cdot|'$ is an ordinal assignment of rank $n+1$ (see Theorem 2.19(i)). Hence, by Theorem 2.6, $\mathrm{RED}_n(|\cdot|')$ is an ordinal assignment of rank n. From the definitions it is clear that $\mathrm{RED}_n(|\cdot|')$ is just the restriction of $\|\cdot\| = \mathrm{RED}_n(|\cdot|_{\mathrm{R}})$ to TR_{n+1}. So it follows that

$$\forall M \in \mathrm{TR}_{n+1}\, [(\forall L \leftarrow M\, \|L\| < \|M\| \vee \exists F \subset_{\mathrm{h}} M\, (\|F\| < \|M\| \wedge \mathrm{Lev}(F) \leqslant n)].$$

In view of the definition of $L < M$ this implies immediately $\forall M \in \mathrm{TR}_{n+1}\, \forall L < M\, \|L\| < \|M\|$ (recall also that $\|\cdot\|$ satisfies (Arg)) and consequently, by the definition of ord,

$$\forall M \in \mathrm{TR}_{n+1}\, \forall L < M\, \mathrm{ord}(\ulcorner L \urcorner) \prec^* \mathrm{ord}(\ulcorner M \urcorner).$$

(ii) This is obvious from a glance at Definition 3.6(iii) and at the clauses (1), (2), ..., (7b) in 2.2. (As to clause (2): if $A \in \mathrm{Var}$, then $A \in \mathrm{FV}(A\vec{K})$ and $\mathrm{Lev}(K_i) < \mathrm{Lev}(A) < \mathrm{Lev}(A\vec{K})^\wedge$.)

(iii) This is routine, by a closer inspection of the constructions in 2.5, 2.7, 2.8, 2.16 and 2.18 (see also Lemma 2.8 and observe, in general, that in the expression for $\mathrm{SUB}_{\mathrm{p}}(|\cdot|)(M)$ in Definition 2.5(i) we may restrict L to the *standard frames* of M (see 2.17(i)), since ordinal assignments identify variants (cf. 2.3). The set of these standard frames is finite for each term M.) As to the functions that represent the relevant ordinal operations with respect to $\prec, \prec \cdot \omega$ and $\prec^*$ (like exponentiation, addition or multiplication with finite ordinals): these belong to $\mathrm{REC}_1(\prec)$ since $\prec$ is a good w.o. (see Lemma 3.2 and also Lemma 3.1(iii)). We omit the details. □

Another crucial property of the modified successor relation is that the value of a term can be computed in a simple, direct way from the values of its (modified) successors. Intuitively this is fairly clear if one looks at the clauses (1), (2), ..., (7b) in 2.2 and (H), (S) in 3.6(iii). Preceding a precise technical formulation of this fact (see Lemma 3.10 below) we first state some necessary, more or less standard facts concerning (1) transformation of functionals to functionals of standard types and (2) coding assignments (in the sense of 1.4) into functionals of appropriate types.

3.9. Notation. $\mathscr{A}$ is the set of all assignments ρ such that

$$\forall x \in \mathrm{dom}(\rho)\, \mathrm{Lev}(x) \leqslant n-1$$

(cf. 1.4). If M is any term, then $\mathscr{A}(M)$ is the set of all $\rho \in \mathscr{A}$ such that ρ is an assignment for M (i.e. $\mathrm{FV}(M) \subseteq \mathrm{dom}(\rho)$; see 1.4).

Fact. (i) *For each type* σ *with* $\mathrm{Lev}(\sigma)\leqslant n$ *there exist in* EXP (*cf.* 1.2) *so-called* transformation functionals $\mathrm{Tr}_{\sigma,n}\in\sigma\to\mathbf{n}$ *and* $\mathrm{Tr}_{n,\sigma}\in\mathbf{n}\to\sigma$ *such that*

$$\forall\Psi\in\sigma\ \mathrm{Tr}_{n,\sigma}(\mathrm{Tr}_{\sigma,n}(\Psi))=\Psi.$$

Here $\mathbf{n}$ *is the standard type of level n, as defined in* 1.1.

(ii) *There exist* (1) *an operation* $\rho\mapsto\ulcorner\rho\urcorner$ *that assigns to each* $\rho\in\mathcal{A}$ *a functional* $\ulcorner\rho\urcorner$ *of the standard type* $\mathbf{n-1}$ (*the so-called code of the assignment* ρ) *and* (2) *for each type* σ *with* $\mathrm{Lev}(\sigma)\leqslant n-1$, *functionals* $\mathrm{AP}_\sigma\in\mathbf{n-1}\to 0\to\sigma$ (*'application'*) *and* $\mathrm{INS}_\sigma\in\sigma\to 0\to\mathbf{n-1}\to\mathbf{n-1}$ (*'insertion'*), *both belonging to* EXP, *such that* (cf. 1.4):

$$\mathrm{AP}_\sigma(\ulcorner\rho\urcorner,\ulcorner x\urcorner)=\rho(x)\qquad \textit{if } \rho\in\mathcal{A},\ x\in\mathrm{Var}(\sigma)\cap\mathrm{dom}(\rho)$$

$$\mathrm{INS}_\sigma(\Psi,\ulcorner x\urcorner,\ulcorner\rho\urcorner)=\ulcorner\rho[\Psi/x]\urcorner\qquad \textit{if } \rho\in\mathcal{A},\ x\in\mathrm{Var}(\sigma),\ \Psi\in\sigma.$$

Proof. (i) See, e.g., Normann [3, pp. 4–6]; although his construction applies only to standard types σ, i.e. $\sigma=\mathbf{m}$ with $m\leqslant n$, it can easily be generalized by application of certain operations for coding finite sequences of functionals by single functionals. The construction of such (standard) operations can also be found in Normann [ibid.].

(ii) The coding $\rho\mapsto\ulcorner\rho\urcorner$ and the corresponding functionals AP_σ and INS_σ ($\mathrm{Lev}(\sigma)\leqslant n-1$) can be constructed by means of certain standard operations for coding finite sequences of functionals of arbitrary types with levels $\leqslant n-1$ by single functionals of type $\mathbf{n-1}$. After all this is just a matter of routine, without specific difficulties, so we omit the details. □

3.10. Lemma. *Let A be a finite set of types. Then there exists a functional* $V\in\mathrm{REC}_1(<)$ *of type* $(0\to\mathbf{n-1}\to\mathbf{n})\to 0\to\mathbf{n-1}\to\mathbf{n}$ *such that for all terms M with* $\mathrm{Lev}(\hat{M})\leqslant n$ *and* $\mathrm{Typ}^*(M)\subseteq A$ *and for all functionals* Θ *of type* $0\to\mathbf{n-1}\to\mathbf{n}$:

$$[\forall L<M\ \forall\rho\in\mathcal{A}(L)\ \Theta(\ulcorner L\urcorner,\ulcorner\rho\urcorner)=\mathrm{Tr}([\![L]\!]_\rho)]\ \Rightarrow$$
$$\forall\rho\in\mathcal{A}(M)\ V(\Theta,\ulcorner M\urcorner,\ulcorner\rho\urcorner)=\mathrm{Tr}([\![M]\!]_\rho)$$

(*where* $\mathrm{Tr}(\Psi)$ *abbreviates* $\mathrm{Tr}_{\sigma,n}(\Psi)$ *for any functional* Ψ *of type* σ *with level* $\leqslant n$).

Proof. The intended functional $V\in(0\to\mathbf{n-1}\to\mathbf{n})\to 0\to\mathbf{n-1}\to\mathbf{n}$ can be obtained by means of a *definition by cases* within $\mathrm{REC}_1(<)$, where the cases to be considered correspond to the clauses (H), (S) in 3.6(iii) and the clauses (1), (2), . . . , (7b) in 2.2. Some of these clauses give rise to a sequence of subcases; for example: in clause (1) in 2.2,

$$F\to FX_1\cdots X_k,$$

we must consider separately each possible type of the term *F*. However, the point is that the number of these subcases is still finite, since all relevant types are taken from the set *A*, which is finite by assumption.

The fact that any auxiliary functional in the ultimate construction of V belongs to $\mathrm{REC}_1(<)$ is either obvious or follows from Fact 3.9 or from a preceding lemma, like e.g. Lemma 3.8(iii) or Lemma 1.2.

More details about the construction of V can be found in a more extensive preprint, which is obtainable from the author. (Preprint nr. 270, University of Utrecht, Department of Mathematics, December 1982.) However, in essence this is just a matter of routine and those details are not very interesting. □

We are now ready to complete the proof of Theorem 3.3 along the lines as sketched in the introduction of this paper. However, rather than the successor relation $\rightarrow$ itself we will actually make use of the modified successor relation $>$ as defined above. (This little inaccuracy in the introduction is justified by reasons of expository convenience.) The reason for this is that $>$ has also the relevant properties which were referred to in the introduction and that it has in addition the following advantageous property: in order to prove that $>$ is wellfounded on the set of terms with which we will be concerned here below, we need considerably smaller ordinals than in the case of $\rightarrow$. (See also the remarks at the beginning of Section 2, just above the conventions concerning $<$.)

3.11. Proof of Theorems 3.3. Let $\Phi \in \mathrm{REC}_{n+1}(<)$ be given such that Φ has type level $\leqslant n$. By Lemma 1.4 there exists a closed term $F \in \mathrm{TR}_{n+1}$ such that $[\![F]\!] = \Phi$. Fix $k \in \mathbb{N}, k \geqslant 1$, such that $\|F\| < \exp(|<| \cdot k)$; this is possible since $\|F\| < \exp(|<| \cdot \omega)$. Write $<' = 2^{<\cdot k}$; then $<'$ is an initial segment of $<^* = 2^{<\cdot\omega}$ (see Lemma 3.1(iv), (v)).

Let $A = \mathrm{Typ}^*(F)$. Obviously this subset A of Typ is finite. So corresponding to A we have a functional $V \in \mathrm{REC}_1(<)$ as in Lemma 3.10. Using V and the function $\mathrm{ord}: \mathbb{N} \rightarrow \mathbb{N}$ (as defined in 3.6(ii)) we define as follows, by transfinite recursion over $<'$, a functional Θ of type $0 \rightarrow \mathbf{n-1} \rightarrow \mathbf{n}$:

$$\Theta(x) = V([\Theta]_x, x),$$

where (using an ad hoc notation) $[\Theta]_x$ is the following functional of type $0 \rightarrow \mathbf{n-1} \rightarrow \mathbf{n}$:

$$[\Theta]_x(y) = \begin{cases} \Theta(y) & \text{if } \mathrm{ord}(y) <' \mathrm{ord}(x), \\ 0^{\mathbf{n-1}\rightarrow\mathbf{n}} & \text{otherwise.} \end{cases}$$

Claim. (1) $\Theta \in \mathrm{REC}_n(<')$.

(2) $$\forall M \in \mathrm{TR}_{n+1} [(\|M\| < |<'| \wedge \mathrm{Lev}(\hat{M}) \leqslant n \wedge \mathrm{Typ}^*(M) \subseteq A) \\ \rightarrow \forall \rho \in \mathscr{A}(M)\ \Theta(\ulcorner M \urcorner, \ulcorner \rho \urcorner) = \mathrm{Tr}([\![M]\!]_\rho)].$$

Proof of Claim. (1) This is clear since Θ has type level n and $V \in \mathrm{REC}_1(<')$ as well as $\mathrm{ord} \in \mathrm{REC}_1(<')$; see Lemma 3.10, Lemma 3.8(iii) and also Lemma 3.1(ii), which (two times applied) implies $\mathrm{REC}_1(<) \subseteq \mathrm{REC}_1(<')$. (*Remark.* In the strict sense a definition by $\prec'$-recursion is of the form $\Psi(x)=\Omega([\Psi]_{\prec' x}, x)$; see 1.2. However, it is an easy exercise to reduce the above definition of Θ to this specific form, still staying within $\mathrm{REC}_n(\prec')$.)

(2) This can be proved by a straightforward induction on $\|M\|$, using the property of V (see Lemma 3.10) and Lemma 3.8(i), (ii); since $\prec'$ is an initial segment of $\prec^*$ it follows from Lemma 3.8(i) that:

$$\forall M \in \mathrm{TR}_{n+1}\, [\|M\| < |\prec'| \to \forall L < M\ \mathrm{ord}(\ulcorner L \urcorner) \prec' \mathrm{ord}(\ulcorner M \urcorner)]. \qquad \square \text{ Claim}$$

Now let $F \in \mathrm{TR}_{n+1}$ be as above, representing the functional Φ with type level $\leqslant n$. Obviously we have: $\|F\| < |\prec'|$ (since, by the choice of k, $\|F\| < \exp(|\prec| \cdot k) = |\prec'|$), $\mathrm{Lev}(\hat{F}) \leqslant n$ (since $\mathrm{Lev}(\hat{F}) = \mathrm{Lev}(F) \leqslant n$), $\mathrm{Typ}^*(F) \subseteq A$ (trivial, by the choice of A) and $\emptyset \in \mathscr{A}(F)$ (since F is a closed term). Hence, by claim (2),

$$\Theta(\ulcorner F \urcorner, \ulcorner \emptyset \urcorner) = \mathrm{Tr}_{\sigma,n}([\![F]\!]_{\emptyset}) = \mathrm{Tr}_{\sigma,n}([\![F]\!]) = \mathrm{Tr}_{\sigma,n}(\Phi),$$

where σ is the common type of Φ and F. Using claim (1) and Fact 3.9(i) we conclude:

$$\Phi = \mathrm{Tr}_{n,\sigma}(\Theta(\ulcorner F \urcorner, \ulcorner \emptyset \urcorner)) \in \mathrm{REC}_n(\prec'). \qquad \square$$

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